Space Time and Gravity: Revision Notes

This set of notes is a summary of the main formulae and key concepts introduced in the course "Space Time and Gravity". They have been thought as a guide for revising the course, but they *__are not__* a complete summary of the course material.

Students should *learn and understand* the physical meaning and the derivation of the concepts and formulae included in this file as they had been presented during the lectures.

Einstein's sum convention applies throughout the paper.

Basic concepts in General Relativity

- Before General Relativity NEWTONIAN GRAVITY was providing a coherent description of gravity by describing:
  - the acceleration of a body in a gravitational potential $\phi(x)$:
  $$\vec{a} = -\vec{\nabla}\phi(x)$$
  which tells how gravitation influences matter's behaviour;
  - the gravitational potential generated by a mass distribution $\rho(x)$:
  $$\nabla^2\phi(x) = 4\pi G\rho(x)$$
  which tells how a matter distribution determines gravitation.
- 1905 - SPECIAL RELATIVITY: but it is valid only for inertial frames. How does special relativity generalise when we deal with accelerated frames?
- Is there a connection between gravity and accelerated frames? Encoded by the *Weak Equivalence Principle*: "The inertial mass and the gravitational mass of a body do coincide".
- Experiments show that the acceleration felt by a body moving freely in a gravitational field is universal and independent on the particle's properties. Thus gravity is an intrinsic property of spacetime rather than being the effect of a propagating field.

- Gravity has only a relative existence: a person in a small box which is freely falling does not experience any gravitational field. Furthermore we can "fake" the effects of a gravitational field by accelerating a box in empty space.

  ⇒ *EINSTEIN'S EQUIVALENCE PRINCIPLE*: "In small enough regions of spacetime the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments".

- It exists a preferred set of trajectories, named *inertial* or *freely falling*, which are those followed by accelerated particles (*i.e.*, particles subjected only to the action of gravity), and we should attribute the effects of gravity to the curvature of spacetime, namely to the curvature of the trajectories which freely falling particles follow.

  ⇒ The notion of inertial frame is replaced by that of *Freely Falling Frame* or *Locally Inertial Frame*: a frame which is subjected only to the local effects of gravity, namely which is moving freely in the proximity of an object deforming spacetime and sourcing gravity, and that is small enough to make the inhomogeneities of the gravitational field undetectable.

  ⇒ We can rephrase the Einstein's Equivalence Principle as: "At any point in spacetime, in any arbitrary gravitational field, it is possible to choose a Locally Inertial Frame such that in an arbitrary neighbourhood of that frame the laws of Physics assume the form as in an unaccelerated coordinate system in the absence of gravity".

- We can get a measure of spacetime curvature already in the Newtonian theory, by observing the non-local effect of gravity

  - Observe the relative motion of two test particles in the proximity of the Earth. The gravitational potential is $\phi(\vec{x})$. One particle sits in $\vec{x}$ and the other in $\vec{y}$, and they are $\vec{\xi}$ apart: $\vec{y} = \vec{x} + \vec{\xi}$.
  - the variation of the relative velocity of the two particles is given by the formula:
  $$\ddot{\xi}^i \approx K_{ij}\xi^j, \qquad , i, j = 1, 2, 3$$
  where a sum over repeated indexes is understood and where $K_{ij} = \frac{\partial^2 \phi(\vec{x})}{\partial x^i \partial x^j}$.
  - Notice that: $\sum_i K_{ii} \equiv \mathrm{Tr}\big[K\big] = \nabla^2 \phi = 4\pi\rho$

SPECIAL RELATIVITY

Postulates of Special Relativity:

- The fundamental laws of physics have the same form in any inertial reference frame.

- The speed of light in the vacuum is a universal constant.

Lorentz Transformations

Two inertial reference frames K, described by the set of coordinates (t, x, y, z), and K', described by the set of coordinates (t', x', y', z'), moving with relative speed $\vec{v}$ along the x-axes are related by the following set of Lorentz transformations ($c = 1$):

$$\begin{cases} t' = \frac{t-vx}{\sqrt{1-v^2}} \\ x' = \frac{x-vt}{\sqrt{1-v^2}} \\ y' = y \\ z' = z \end{cases} \longrightarrow \begin{cases} t' = \cosh\zeta t - \sinh\zeta x \\ x' = -\sinh\zeta t - \cosh\zeta x \\ y' = y \\ z' = z \end{cases}$$

where we have introduced the rapidity ζ defined by $v = \tanh\zeta$.

- The Lorentz transformation leave invariant the spacetime interval:

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

- $(\Delta s)^2$ is the distance between two points (events) in Minkowsky spacetime.
- If we consider two points in Minkowsky spacetime P in (t_P, x_P, y_P, z_P) and Q in $(t_P + dt, x_P + dx, y_P + dy, z_P + dz)$, the (infinitesimal) invariant distance between P and Q is:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

- The (finite or infinitesimal) distance between two events in spacetime can be greater, less, or equal to zero. According to this, the two events are:
  - if $(\Delta s)^2 < 0 \Rightarrow$ *timelike* separated $\Rightarrow$ this for example is the case of events happening at the same place but at different time; can be connected by a particle travelling at speed lower than the speed of light;
  - if $(\Delta s)^2 = 0 \Rightarrow$ *lightlike* (or *null*) separated $\Rightarrow$ can be connected by light rays;
  - if $(\Delta s)^2 > 0 \Rightarrow$ *spacelike* separated $\Rightarrow$ this for example is the case of events happening at the same place but at different time; cannot be connected by a particle moving at speed lower or equal to the speed of light.
- The *light cone* of a point P is the locus of points which are null separated from P. Each point P in spacetime has its own light cone.
  - The *future* light cone is generated by the light rays moving outward from P.

- The *past* light cone is generate by the light rays converging in P.
- The points that are timelike separated from P lay inside the light cone. The points that are spacelike separated from P lay outside the light cone.
- Particles with non zero rest mass move along timelike paths which are always inside the light cones at each point along their trajectories.

- The proper time τ is defined by:

$$d\tau^2 = -ds^2 \qquad (d\tau^2 = -\frac{ds^2}{c^2} \,\text{if}\, c \neq 1)$$

and can be defined as the time that would be measured by a clock carried by the observer himself.

- Let's introduce the four vector $x^\mu = (t, x, y, z)$, where μ is an index taking values from 0 to 3 (namely $x^0 = t$, $x^1 = x$, etc.) . Then we can define the scalar product (Einstein sum convention applies):

$$x \cdot x = x^\mu \eta_{\mu\nu} x^\nu = -t^2 + x^2 + y^2 + z^2$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \longleftarrow \qquad \begin{array}{c} \text{is the} \\ \textbf{Minkowsky metric} \end{array}$$

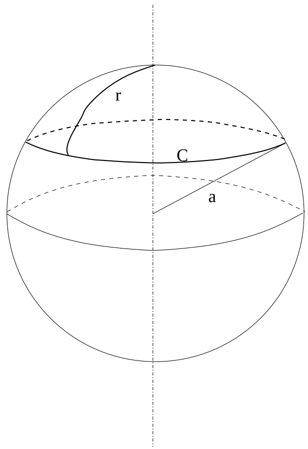
- The infinitesimal Minkowsky spacetime separation is rewritten as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

CURVED GEOMETRIES

- Can be described through their embedding in higher dimensional spaces. Example: a two-dimensional spherical surface can be seen as the set of points in $\mathbb{R}_3$ whose coordinates satisfy the equation $x^2 + y^2 + z^2 = r^2$.
- Can otherwise be described through a set of intrinsic properties, which do not depend on anything outside the surface itself. These are:
  - The set of *geodesics* of a surface, namely the paths of minimal distance between two points in the curved space under consideration. They are the equivalent of straight lines of Euclidean geometry, because they are the "straightest" possible path one can draw on the surface.
    Example: On the 2-dimensional sphere, the geodesics are half great circles.

– The lenght of segments (portions of geodesics) on a given spacetime, and their relations provide a way of deriving its *intrinsic curvature.* Example: We know from Euclidean geometry that the ration between the circumference of a circle drawn on a flat surface and its radius is $\frac{C}{r} = 2\pi$. On the other hand, if we draw a circle of radius r on the surface of a 2-dimensional sphere (see figure), we get:



$$\frac{C}{r} = 2\pi \frac{sin\frac{r}{a}}{\frac{r}{a}}$$

Expanding the relation above for small values of r we get:

$$\frac{C}{r} \approx 2\pi(1 - \frac{1}{6}Kr^2)$$

where $K = \frac{1}{a^2}$ is the *(intrinsic) Gaussian curvature* of a 2 dimensional surface in the neighbourhood of a point P.

– A general local definition for the intrinsic curvature of a 2-dimensional surface is then:

$$K = \frac{3}{\pi} \lim_{r \to 0} \frac{2\pi r - C}{r^3}$$

– Besides the circumference C, other geometrical properties of a spacetime which depend on its intrinsic curvature are:
  * the *area* A of a circle on a 2-dimensional surface:

$$A \approx \pi r^2(1 - K\frac{r^2}{12} + \ldots)$$

  * The *second variation of the spread* between two geodesics intersecting at a point P:

$$\ddot{\eta} = -K\eta$$

These properties depend on the sign of K:

  * if $K > 0$, C and A are smaller than the flat space case, and two geodesic spread with decreasing speed;
  * if $K < 0$, C and A are bigger than the flat space case, and two geodesic spread with increasing speed;
  * if $K = 0$, we retrieve the flat space results.

Metric

The metric provides a description of a given spacetime geometry which is not restrained to the neighbourhood of a point.

In an arbitrary 4-dimensional curved geometry the line element will be a generalisation of the Minkowsky metric. In general, we define:

(n-dimensional) ***Riemannian Space***: is an n-dimensional space parametrised by a set of n coordinates x^μ $(\mu = 0, \ldots, n-1)$, whose intrinsic geometry is defined by a metric $g_{\mu\nu}(x)$ such that:

$$ds^2 = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu \qquad \text{and} \qquad \det g \neq 0$$

Important remarks:

- The inverse metric $g^{\mu\nu}$ corresponds to the inverse matrix: $g_{\mu\rho} g^{\rho\nu} = \delta^\rho_\mu$.
  Example: A 2-dimensional Riemannian space is described by the line element $ds^2 = x^2 dx^2 + xy dy^2$. The metric is $g_{\mu\nu} = \begin{pmatrix} x^2 & 0 \\ 0 & xy \end{pmatrix}$. The inverse metric is $g_{\mu\nu} = \begin{pmatrix} x^{-2} & 0 \\ 0 & (xy)^{-1} \end{pmatrix}$.

- The set of coordinates x^μ is not necessary uniquely defined. One can have different coordinate systems related by a non-degenerate change of coordinates, and the form of the line element will not in general be preserved when switching from one set of coordinates to another.
  Example: Let's consider the Euclidean plane $\mathbb{R}^2$:

  - in Cartesian coordinate (x, y):

  $$ds^2 = dx^2 + dy^2 \qquad \Rightarrow \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - in polar coordinates (r, θ)

  $$ds^2 = dr^2 + r^2 d\theta^2 \qquad \Rightarrow \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

- It may happen that a single coordinate set is not sufficient to describe the full space. We can encounter a *coordinate singularity*. This is a point P, of coordinates x_P such that one or more elements of the metric diverges for $x = x_P$, but such that the curvature scalars behave well in that point.
  Example: The metric $ds^2 = \frac{1}{\rho^4}(d\rho^2 + \rho^2 d\theta^2)$ has a singularity for $\rho = 0$. Is this a true curvature singularity? Does the geometry behaves badly for $\rho = 0$? If we perform the change of coordinates $\rho = \frac{1}{r}$ the metric becomes $ds^2 = dr^2 + r^2 d\theta^2$. This is just the metric on the Euclidean plane in polar coordinates. The Euclidean plane is flat everywhere, the point $\rho = 0$ corresponds to all the set of points at infinity which are taken to a single point by the change of coordinates $r \to \rho$. This simply means that the set of coordinates (ρ, θ) are not suitable to describe this point.
  Another example of coordinate singularity is the Schwarzschild radius (see later).

- If a geometry can be described through its embedding into a higher dimensional space, its metric can be derived restraining the line element of this space on the lower dimensional one.
  Example: S^2 can be embedded in R^3
  - R^3 line element: $ds^2 = dx^2 + dy^2 + dz^2$;
  - each point on S^2 can be parametrised by two angular coordinates (θ, ϕ) such that $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, and $z = R\cos\theta$. Implementing this in the line element above yields to the metric on S^2: $ds^2|_{S_2} = R^2(d\theta^2 + \sin^2\theta d\phi^2)$.
- Spacetime is a 4-dimensional Riemannian space.
  - The equivalence principle suggests that *locally* the properties of spacetime should be indistinguishable from those of the flat spacetime of special relativity. This means that given a metric $g_{\mu\nu}(x)$ at each point P of spacetime it is possible to introduce a new set of coordinates $x'^{\mu}(x)$ such that
$$g'_{\mu\nu}(x'^{\alpha}_P) = \eta_{\mu\nu}$$
    where $\eta_{\mu\nu}$ is the flat Minkowsky metric and x'^{α}_P are the coordinates of the point P.
  - Furthermore one can find a change of coordinates such that *locally* in a point P:
$$g'_{\mu\nu}(x'^{\alpha}_P) = \eta_{\mu\nu} \qquad \text{and} \qquad \left.\frac{\partial g'_{\mu\nu}}{\partial x'^{\rho}}\right|_P = 0$$
  - This is the mathematical description of a *freely falling frame* or *locally inertial frame*.
  - If spacetime is curved, it is *not* possible to find a change of coordinates such that the metric becomes flat *everywhere*. This can happen only if the whole spacetime is flat, and there is no curvature (there is no gravity $\rightarrow$ gravity is spacetime curvature).
- The definition of *timelike*, *spacelike* and *lightlike* separations hold in curved space as well. Massive particles move along timelike trajectories, and the set of lightlike curves connected with a point P form the light cone in P.
- Given a metric $ds^2 = g_{tt}dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$
  - the infinitesimal proper lenght along the directions t, x, y, and z are respectively $\sqrt{g_{tt}}dt$, $\sqrt{g_{xx}}dx$, $\sqrt{g_{yy}}dy$, and $\sqrt{g_{zz}}dz$.
  - the area element along the directions x and y is $dA = \sqrt{g_{xx}g_{yy}}dxdy$; similar formulae hold for infinitesimal area elements along different directions;
  - the infinitesimal 3-volume element is $dV = \sqrt{g_{xx}g_{yy}g_{zz}}dxdydz$;
  - the infinitesimal 4-volume element is $dW = \sqrt{-g_{tt}g_{xx}g_{yy}g_{zz}}dtdxdydz$;

Tensor calculus

Principle of General Covariance

The laws of Physics can be stated in a form which is independent on the specific choice of spacetime coordinates.

The laws of Physics must be valid in any system of coordinates. Thus, they must be expressible as tensor equation. In this way, they are *covariant*: they retain the same form upon a coordinate transformation.

Coordinate transformations

Let's focus on a d-dimensional spacetime geometry and let's consider an invertible coordinate transformation:

$$x^\mu \to x'^\mu, \qquad \mu = 0, \ldots, d-1$$

.

Under such a coordinate transformation:

- a *scalar* field is invariant:
$$\phi(x) = \phi'(x')$$

- a *contravariant vector* transforms as:
$$A^\mu(x) \longrightarrow A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$$
where $\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$;

- a *covariant vector* transforms as:
$$B_\mu(x) \longrightarrow B'_\mu(x') = \Lambda_\mu{}^\nu B_\nu(x)$$
where $\Lambda_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}$.

The contraction of a contravariant and a covariant vector field is a scalar under a coordinate transformation, hence: $\Lambda_\mu{}^\nu = (\Lambda^\mu{}_\nu)^{-1}$.

Tensors

A *tensor* $T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}$ is a multi-index mathematical object which transforms linearly and homogeneously under a spacetime coordinate transformation:

- every *contravariant* (up) index transforms with $\Lambda^\mu{}_\nu$
- every *covariant* (down) index transforms with $\Lambda_\mu{}^\nu$

Remarks.

- The total number of indexes denotes the rank of the tensor. Hence:

- $T^{\mu_1\mu_2\ldots\mu_p}$ is a contravariant tensor of rank p;
- $T_{\mu_1\mu_2\ldots\mu_q}$ is a covariant tensor of rank q;
- $T^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}$ is a mixed tensor of rank $p+q$;

- The null tensor (every component equal to zero) is zero in any reference frame.

- The sum of two tensor of one kind is still a tensor of the same kind:

$$A_\mu(x) + B_\mu(x) = C_\mu(x)$$

- The product of two tensors is a tensor with rank equal to the sum of the ranks of the two factors:

$$A^{\mu_1\mu_2\ldots\mu_p} \cdot B_{\nu_1\nu_2\ldots\nu_q} = C^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}$$

- The metric is a covariant tensor of rank 2. Under a coordinate transformation it transforms as:

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}(x)$$

- Contraction of indexes:
  - contracting a covariant and a contravariant tensor we obtain a scalar (*e.g* $A_\mu B^\mu = A_0B^0 + A_1B^1 + A_2B^2 + A_3B^3$ in 4 dimensions);
  - we can also contract covariant and contravariant indexes in the same tensor: $T^\mu{}_\mu \equiv T = T^0{}_0 + T^1{}_1 + T^2{}_2 + T^3{}_3$;
  - this generalises to the contraction of any pair of tensor indexes: $T^{\rho\mu}{}_\mu \equiv T^\rho$ is a contravariant tensor of rank 1, $T_{\mu\nu\sigma}{}^{\mu\nu} \equiv T_\sigma$ is a covariant tensor of rank 1, etc.

- we can use the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ to raise and lower the indexes:
  - $T_\mu = g_{\mu\nu}T^\nu$
  - $T^\mu = g^{\mu\nu}T_\nu$
  - $T^\mu{}_\mu = T^{\mu\nu}g_{\mu\nu} = T_{\mu\nu}g^{\mu\nu}$

The Covariant Derivative

The ordinary derivative of a scalar is a tensor, but the derivative of a covariant (or contravariant) vector is not. In fact, it does not transforms covariantly as a tensor under a coordinate transformation $x^\mu \to x'^\mu$:

$$\partial_\mu \phi(x) \xrightarrow{x\to x'} \Lambda_\mu{}^\nu \partial_\nu \phi(x) \qquad \checkmark$$

$$\partial_\mu V_\nu(x) \xrightarrow{x\to x'} \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \partial_\rho V_\sigma(x) + \frac{\partial^2 x^\rho}{\partial x'^\mu \partial x'^\nu} V_\rho(x) \qquad \times$$

We overcome this problem by introducing the *covariant derivative*:

$$D_\mu V_\nu(x) = \partial_\mu V_\nu(x) - \Gamma^\alpha_{\mu\nu} V_\alpha(x)$$

where the set of coefficient $\Gamma^\alpha_{\mu\nu}$ form the *connection*. The connection is not a tensor: under a coordinate transformation, it transforms as:

$$\Gamma^\alpha_{\beta\gamma} \xrightarrow{x \to x'} \Gamma'^\alpha_{\beta\gamma} = \Lambda^\alpha{}_\mu \Lambda_\beta{}^\nu \Lambda_\gamma{}^\rho \Gamma^\mu_{\nu\rho} + \Lambda_\rho{}^\alpha \frac{\partial^2 x'^\rho}{\partial x'^\beta \partial x'^\gamma}$$

The covariant derivative of a covariant vector $D_\mu V_\nu(x)$ transforms as a covariant rank-2 tensor.

We define similarly the covariant derivative of a contravariant vector:

$$D_\mu B^\nu(x) = \partial_\mu B^\nu(x) + \Gamma^\nu_{\mu\rho} B^\rho(x)$$

Under a coordinate transformation $D_\mu B^\nu(x)$ transforms as a mixed tensor of rank 2: $D_\mu B^\nu(x) \xrightarrow{x \to x'} D'_\mu B'^\nu(x) = \Lambda_\mu{}^\rho \Lambda^\nu{}_\sigma D_\rho B^\sigma(x)$

These definitions are easily generalised to tensors of arbitrary rank.

Remarks.

- The covariant derivative is distributive: $D_\mu(TS) = (D_\mu T)S + T(D_\mu S)$ (T and S are tensors of arbitrary rank).
- The covariant derivative of scalar is equivalent to the ordinary derivative: $D_\mu \phi(x) = \partial_\mu \phi(x)$
- Demanding that the covariant derivative commute with the operation of raising and lowering indexes, $D_\mu(A_\rho g^{\rho\nu}) \equiv (D_\mu A_\rho) g^{\rho\nu}$, we derive the further condition:

  $$D_\mu g^{\rho\nu} = 0$$ *The covariant derivative of the metric is identically zero*

  This last condition can be expanded and solved for the connection coefficient, yielding to the definition of the *Christoffel Symbols*:

  $$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(\partial_\beta g_{\mu\gamma} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma})$$

- We define the *directional covariant derivative* along a path $x^\mu(s)$ as:

  $$\frac{D}{Ds} = \frac{dx^\mu}{ds} D_\mu.$$

  The directional covariant derivative does not change the rank of a tensor, it just moves it along the path!
  A tensor $T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q}$ is parallely transported along the path $x^\mu(s)$ if its directional covariant derivative along that path is zero: $\frac{dx^\rho}{ds} D_\rho T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = 0$

Geodesics Equations

Geodesics are the equivalent in curved space of what straight lines are in flat space. They can then be defined as the curves whose tangent vector does not vary when moved along: *the Geodesics of a curved space are the set of curves along which the tangent vector is parallely transported.*

These are the set of curves satisfying the equations:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

In a given geometry, let's consider a curve parametrised by $x^\mu(s)$. The vector tangent to the curve is $u^\mu = \frac{dx^\mu(s)}{ds}$, and it is constant along the curve if $\frac{du^\mu}{ds} = 0$. But :

$$0 = \frac{du^\mu}{ds} = \frac{dx^\rho}{ds}\partial_\rho u^\mu \quad \xrightarrow{\text{COVARIANTISE}} \quad \frac{dx^\rho}{ds}D_\rho u^\mu = 0$$

We obtain the geodesics equations by expanding the expression for the covariant derivative in the last formula.
Remarks.

- Geodesics are the trajectories followed by unaccelerated particles, subjected only to the action of gravity.
- In flat space and in Cartesian rectilinear coordinates $\Gamma^\mu_{\nu\rho} = 0$, and the geodesics equations reduce to $\frac{d^2x^\mu}{ds^2} = 0$, *i.e.* they describe the motion of a particle on a straight line.
- For timelike trajectories the parameter s can be any affine parameter, namely any parameter related to the proper time τ by a linear transformation: $s = a\tau + b$, a and b constants.
- It is possible to derive the geodesic equations as the equations of the trajectories extremising the proper time between the starting point A and end point B of the particle trajectory:

$$\tau_{AB} = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}}ds$$

- Along the geodesics the following normalisation condition holds:

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = constant$$

  If the affine parameter s coincides with the proper time τ, then the constant is equal to -1.

Locally Inertial Frames

Let's consider a Riemannian spacetime whose geometry is described by the metric $g_{\mu\nu}(x)$, in the set of coordinates x^μ. Then, at any point P it is possible to introduce a new set of coordinates, namely a new reference frame, which we denote x'^μ, such that:

$$g'_{\mu\nu}(x'_P) = \eta_{\mu\nu} \qquad \text{and} \qquad \frac{\partial g'_{\mu\nu}}{\partial x'^\rho}\Big|_P = 0$$

where $\eta_{\mu\nu}$ is the Minkowsky metric and x'_P are the coordinates of P in the new set.

This is the formal definition of a Locally Inertial Frame.

Remarks:

- The condition $\frac{\partial g'_{\mu\nu}}{\partial x'^\rho}|_P = 0$ is automatically satisfied if, in the new set of coordinates, $\Gamma'^\mu_{\nu\rho}|_P = 0$.
- These relations hold locally in a small neighbourhood of a point P: one cannot find a unique coordinate transformation under which they hold everywhere, unless the geometry under consideration is flat.
- The second derivatives of the metric are non-zero and they enter in the definition of the spacetime *intrinsic* curvature in the neighbourhood of the point P.
- The spacetime distance between the point P and its neighbouring points can be either computed in the old set of coordinates or in the LIF coordinates, and they can be either negative, positive, or null. As usual light rays move along null directions, while massive particles along timelike worldlines. Locally, light-cone structure is the same as in special relativity, it is the distribution of light cones in spacetime which varies.

The Riemann Curvature Tensor

Parallel transport between two points along two different paths yields to two different results. This is related to the fact that we cannot exchange the action of two subsequent covariant derivatives. In a curved geometry the commutator of two covariant derivatives acting on a four-vector A_μ is non zero, and quantifies the spacetime curvature:

$$D_\sigma D_\nu A_\mu - D_\sigma D_\nu A_\mu = A_\rho R^\rho{}_{\mu\nu\sigma}$$

The tensor $R^\rho{}_{\mu\nu\sigma}$ is the *Riemann Curvature Tensor*:

$$R^\rho{}_{\mu\nu\sigma} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\rho_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\rho_{\alpha\sigma}.$$

Remarks.

- The components of the Riemann tensor depend on the second derivatives of the metric: in a curved geometry they cannot be all put to zero by changing to Riemann Normal Coordinates;
- If, in a given spacetime, it exists a coordinate choice for which $R^{\rho}{}_{\mu\nu\sigma} = 0$ for every value of the indexes, then that spacetime is flat.
- Symmetries of the Riemann Tensor:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$$

  where $R_{\mu\nu\rho\sigma} = g_{\mu\alpha}R^{\alpha}{}_{\nu\rho\sigma}$

Ricci Tensor

Obtained by contracting the first and third indexes of the Riemann Tensor:

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$$

Ricci Scalar

Obtained by contracting the Ricci tensor with the metric:

$$R = g^{\mu\nu}R_{\mu\nu}$$

Einstein Tensor

It is the combination:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

Bianchi Identities

$$D_{\lambda}R_{\rho\sigma\mu\nu} + D_{\rho}R_{\sigma\lambda\mu\nu} + D_{\sigma}R_{\lambda\rho\mu\nu} = 0$$

- The Bianchi Identities can be verified by rewriting $D_{\lambda}R_{\rho\sigma\mu\nu}$ in Riemann Normal Coordinates (such that $\Gamma = 0$ but $\partial\Gamma \neq 0$), and then summing over the cyclical permutations of λ, ρ and σ.
- If we contract the Bianchi identities written above with $g^{\rho\sigma}g^{\mu\nu}$ we get:

$$D^{\mu}G_{\mu\lambda} = D^{\mu}\Big(R_{\mu\lambda} - \frac{1}{2}Rg_{\mu\lambda}\Big) = 0$$

  The Einstein tensor is conserved.

Equation of Geodesic Deviation

It describes the relative acceleration between two geodesics $x^{\mu}(s)$ and $x^{*\mu}(s) = x^{\mu}(s) + \xi^{\mu}(s)$:

$$\frac{D^2\xi^{\mu}}{Ds^2} = -R^{\mu}_{\nu\rho\sigma}\xi^{\rho}\frac{dx^{\nu}}{ds}\frac{dx^{\sigma}}{ds}$$

The parameter s is a generic affine parameter along the geodesics.

Minimal Coupling Between Matter and Gravity

Gravity is universal: it can be described as a feature of the background where particles move rather than as a propagating field:

- free particles are those subjected only to the effects of gravity;
- free particles move along geodesics defined by the geodesic equation.

The minimal coupling between matter and gravity let us combine these findings with the laws of physics that we gather from special relativity,

To minimally couple matter and gravity we:

1. take the laws of physics valid in an inertial frame in Minkowsky spacetime;
2. write them in a covariant form;
3. assert that the results which they describe remain valid in a curved geometry characterised by the metric $g_{\mu\nu}$, then we get their curved space counterpart by replacing:
    - $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$
    - $\partial_\mu \longrightarrow D_\mu$

Examples.

- <u>GEODESIC EQUATIONS</u> The minimal coupling is exactly what we did when we derived the equations for the geodesics. We:
    - wrote in a covariant form the equation demanding that the tangent vector is constant along the geodesics: $\frac{dx^\rho}{ds}\partial_\rho u^\mu$;
    - replaced the ordinary derivative ∂_ρ with the covariant derivative D_ρ.
- <u>MAXWELL EQUATIONS</u> In covariant form in Minkowsky spacetime they read $\partial_\mu F^{\mu\nu} = -j^\nu$. Minimally coupled with gravity, they become:

$$D_\mu F^{\mu\nu} = -j^\nu$$

  where all the contractions are performed with the metric $g_{\mu\nu}$.

- <u>CONSERVATION LAWS</u>. In a covariant form and in flat spacetime the conservations of energy and momentum is encoded in the conservation of the stress energy tensor $T^{\mu\nu}$. Then:

$$\partial_\mu T^{\mu\nu} = 0 \quad \xrightarrow{min.coupling} \quad D_\mu T^{\mu\nu} = 0$$

Newtonian limit of GR

We must recover the results of Newtonian gravity under the following conditions:

1. The particles move slowly with respect to the speed of light.
2. Gravity is weak, and can be treated as a perturbation of flat spacetime.
3. Fields are static: no dependence on the coordinate time t.

Newtonian limit of the geodesics equations

Let's focus on the trajectories of timelike particles, which we parametrise with their proper time τ. At the level of the geodesic equation, the three conditions above are implemented by requiring:

1. $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \qquad \forall\, i = 1, 2, 3$
2. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with}\, |h_{\mu\nu}| << 1$
3. $\frac{\partial}{\partial t} g_{\mu\nu} = 0 \qquad \forall\, \mu, \nu = 0, 1, 2, 3.$

Implementing this condition in the geodesic equation we get:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \qquad \text{if}\, \mu = 0 \implies \qquad \frac{dt}{d\tau} = const.$$

$$\text{if}\, \mu = i \neq 0 \implies \qquad \frac{d^2x^i}{dt^2} = \frac{1}{2}\partial^i h_{00}$$

- The first relation tells us that the coordinate time can be used as parameter along the geodesics, while the second relation shows that, upon identifying $h_{00} \equiv -2\phi(\vec{x})$, where $\phi(\vec{x})$ is the Newtonian potential, we get exactly the Newton equation for gravity:

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi(\vec{(x)})$$

- The identification between h_{00} and ϕ holds if we can write $g_{00} = -(1 + 2\phi(x))$, namely if this component of the metric is a solution of the Einstein equations (the equations that determine the explicit form of the metric).

Einstein Equations

- The Einstein equations are the relativistic generalisation of the Poisson equation for the Newton potential $\nabla^2\phi = 4\pi G\rho(x)$, where $\rho(x)$ is the mass density.
- They have the form:

TENSOR DESCRIBING THE SPACETIME GEOMETRY	=	TENSOR DESCRIBING THE SPACETIME ENERGY CONTENT

The Stress-Energy tensor

This is the tensor describing the matter and energy content of spacetime.

- Physical definition:

$$T^{\mu\nu} \quad \Longrightarrow \quad \begin{array}{c}\text{Flux of four-momentum } p^\mu \\ \text{across a surface of constant } x^\mu\end{array}$$

- Separating the "0" coordinate from "i" $= 1, 2, 3$, we identify:

$$\left(\begin{array}{c|c} T^{00} & T^{0j} \\ \hline T^{i0} & T^{ij} \end{array}\right)$$

  where

  - T^{00}: mass-energy density;
  - T^{i0}: flux of energy;
  - T^{0j}: momentum density;
  - T^{ij}: momentum flux: stress

- For a perfect fluid

$$T^{\mu\nu}_{\text{fluid}} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}$$

  where

  - ρ is the mass-energy density of the fluid;
  - p is the pressure (same in every direction);
  - U^μ is the average macroscopic 4-velocity.

- The stress-energy tensor is conserved: $D_\mu T^{\mu\nu} = 0$. If we integrate this relation over a volume V we recover the equations for the conservation of energy (integrating the relation for $\nu = 0$) and for the conservation of the three components of the linear momentum (integrating the three relations for $\nu = i \neq 0$).

Complete form of the Einstein Equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

where:

→ $g_{\mu\nu}$ is the metric describing the geometry of a given spacetime.

→ $R_{\mu\nu} = R_{\alpha\mu\beta\nu}g^{\alpha\beta}$ is the Ricci tensor.

→ $R = R_{\mu\nu}g^{\mu\nu}$ is the Ricci scalar.

→ $T_{\mu\nu}$ is the stress-energy tensor, which describes the matter and energy content of a given spacetime.

→ Λ is the cosmological constant, which describes the vacuum energy density, namely the energy density of empty space.

Einstein equations in the vacuum

In the vacuum $T_{\mu\nu} = 0$, then the Einstein equations reduce to:

$$R_{\mu\nu} = 0$$

Remarks.

- These equations can be derived comparing the equation for the geodesic deviation with the equation for the tidal forces in the case when the mass density of the gravitational source is zero.
- In the static and weak field limit they reduce to the Poisson equation in empty space:
$$R_{\mu\nu} \sim -\delta_{\mu\nu}\nabla^2\phi(x)$$
then $R_{00} = 0 \Rightarrow \nabla^2\phi(x) = 0$.

The Schwarzschild Metric

It is the unique static and spherically symmetric solution of the Einstein equations in the vacuum $R_{\mu\nu} = 0$. In spherical coordinates (t, r, θ, ϕ):

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where M is the mass sourcing the gravitational field and G is the universal gravitational constant.

- The coefficients of the metric becomes singular for $r = 0$ and $r = 2GM$. The quantity $r_S \equiv 2GM$ is called Schwarzschild radius and it is a coordinate singularity of the Schwarzschild metric (if $c \neq 1$ $r_S \equiv \frac{2GM}{c^2}$). When $r = 2GM$ the coefficient g_{rr} of the metric becomes singular, however this is due just to the particular coordinate set we are using, and the geometry behaves well. To prove it one must consider scalars built out of the curvature Riemann tensor, and check if they are finite or if they diverge in the singular points of the metric. For example one can consider
$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48G^2M^2}{r^6}$$
We see that $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \xrightarrow{r\to 0} \infty$, so $r = 0$ is a true curvature singularity, while $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}|_{R=2GM} = \frac{3}{4G^4M^4}$ is finite at the Schwarzschild radius.

- The Schwarzschild radius has a physical meaning only for ultra-compact objects, such that this distance is still in the vacuum, and the neighbouring geometry is still described by the Schwarzschild metric (*e.g.*, the Schwarzschild radius of the Sun $\frac{2GM_\odot}{c^2}$ is about 2.9km, well inside the surface of the star, where the geometry is no longer described by the Schwarzschild solution).

- The Schwarzschild radius is the distance of the *event horizon* of a black hole.

- The *weak field limit* of the Schwarzschild metric holds far from the mass sourcing the geometry, namely for $r \gg 2GM$. At such distances the metric can be seen as a perturbation of the Minkowsky metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. But we can rewrite $g_{00} = -1 + \frac{2GM}{r} = \eta_{00} - 2\phi(r)$, where $\phi(r)$ is the Newtonian potential. Similarly, if we take the weak field limit of g_{rr} we get:
$$g_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1} \xrightarrow{r \gg 2GM} 1 + \frac{2GM}{r} = \eta_{11} - 2\phi(r)$$
  and the metric becomes:
$$ds^2 \xrightarrow{r \gg 2GM} -(1 + 2\phi(r))dt^2 + (1 - 2\phi(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
  To write this as a perturbation of Minkowsky, we can choose $h_{\mu\nu} = -2\phi\delta_{\mu\nu}$.

- The metric tends to Minkowsky for $r \to \infty$. This phenomenon is called *asymptotic flatness*, and it is a consequence of the metric, not an ansatz.

- The metric tends to Minkowsky for $M \to 0$ (no mass sourcing the geometry).

Tests of General Relativity

Three phenomena predicted by General Relativity which can be considered as experimental tests of the theory are:

- The precession of planet's perihelia
- The bending of light rays in proximity of a gravitational source
- The gravitational redshift

To quantify the effects of the first two phenomena we need to know the orbits of test particles in the Schwarzschild geometry. Spherical symmetry constraints the orbits to lay on planes of constant θ. Let's fix $\theta = \frac{\pi}{2}$.

Orbits of massive particles

- Timelike orbits can be computed by exploiting the following conditions:
  - Conservation of energy (because of no time dependence):

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = constant = k$$

  - Conservation of angular momentum (because of spherical symmetry):

$$r^2\frac{d\phi}{d\tau} = constant = l$$

  - normalisation condition along timelike geodesics $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1$:

$$\left(1 - \frac{2GM}{r}\right)\frac{d^2t}{d\tau^2} - \left(1 - \frac{2GM}{r}\right)^{-1}\frac{d^2r}{d\tau^2} - r^2\frac{d^2\phi}{d\tau^2} = 1$$

- Combining these three conditions we find that particles move along the orbits of the effective potential

$$V_{eff}(r) = -\frac{MG}{r} + \frac{l^2}{2r^2} - \frac{l^2MG}{r^3}$$

- We have bigger discrepancy with respect to the Newtonian effective potential $V_N(r) = -\frac{MG}{r} + \frac{l^2}{2r^2}$ for smaller values of r: search for measurable effects in the orbits of innermost planets of the Solar System.

Precession of planets' perihelia

- Introducing $u = \frac{1}{r}$, the variation of u with r is encoded by the equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{MG}{l^2} + 3MGu^2$$

  while the Newtonian effective potential gives: $\frac{d^2u}{d\phi^2} + u = \frac{MG}{l^2}$.

- The extra term is small for solar system planets, and can be treated as a perturbation, leading to the approximate solution

$$u_{GR} = \frac{GM}{l^2}\left[1 + e\cos\left[\phi\left(1 - \frac{3G^2M^2}{l^2}\right)\right]\right]$$

- u_{GR} is a periodic function of period $\frac{2\pi}{1-\frac{3G^2M^2}{l^2}} > 2\pi$.

- For Mercury $\frac{2\pi}{1-\frac{3G^2M^2}{l^2}} - 2\pi \sim 42.9$" per century.

Photons' orbits and deflection of light rays

- Lightlike trajectories can be worked out exactly in the same way as for timelike, but now considering that for lightlike trajectories $ds^2 = 0$, then the normalisation condition along the geodesic reads:

$$\left(1 - \frac{2GM}{r}\right)\frac{d^2t}{d\tau^2} - \left(1 - \frac{2GM}{r}\right)^{-1}\frac{d^2r}{d\tau^2} - r^2\frac{d^2\phi}{d\tau^2} = 0$$

- The equation for $u(\phi)$ now reads:

$$\frac{d^2u}{d\phi^2} + u = 3MGu^2$$

- If $M = 0$ the equation this is the equation for a straight line
- When $M \neq 0$ it is solved by:

$$\frac{1}{r} = \frac{\sin\phi}{R} + \frac{MG}{2R}(3 + \cos 2\phi)$$

  where R is an integration constant.

- in Cartesian coordinates the distance from the gravitational source along the y axes varies as:

$$y = R - \frac{GM}{R}\frac{y^2 + 2x^2}{\sqrt{x^2 + y^2}},$$

  and the total deflection from $x \to -\infty$ to $x \to +\infty$ is equal to

$$\Theta \sim -2\frac{y}{x}\Big|_{x\to+\infty} \sim \frac{4GM}{R}$$

  (if $c \neq 1 \Rightarrow \frac{4GM}{Rc^2}$)

- The deflection of starlight by the Sun is about 1.75".

Gravitational redshift

Light climbing a gravitational field gets redshifted.
Let's consider an atom in a gravitational field sitting at a distance r_e, which emits radiation with frequency ν_e. The emitted pulse travels radially towards an observer sitting at a distance r_0. If the metric is static (*i.e.*, it does not depend on time), the observed radiation will have frequency ν_o such that:

$$\frac{\nu_0}{\nu_e} = \sqrt{\frac{g_{00}(r_e)}{g_{00}(r_o)}}$$

where g_{00} is the coefficient of dt^2 in the metric.

In Schwarzschild geometry:

$$\frac{\nu_0}{\nu_e} = \sqrt{\frac{1 - \frac{2GM}{r_e}}{1 - \frac{2GM}{r_o}}}$$

Remarks.

- To derive the formula for the gravitational redshift one must explicitly use the fact that the metric does not depend on the coordinate time t, and then in coordinate time the period of the emitted and of the observed wave is the same.
- If we are far from the source ($r \gg 2GM$), and $\frac{\nu_o}{\nu_e} \approx 1 - \frac{GM}{r_e} + \frac{GM}{r_o} = 1 + \phi(r_e) - \phi(r_o)$, where $\phi(r)$ is the Newton potential. Since $\phi(r_e) > \phi(r_0)$, $\nu_o < \nu_e$, *i.e.*, the radiation gets redshifted.

Effective speed of light

The curvature induced around a massive body increases the travel time of light rays relative to what it would be in flat space. For timelike geodesic $ds^2 = 0$, and if we focus on radial trajectories $d\phi = d\theta = 0$, then:

$$\Big(1 - \frac{2GM}{r}\Big)dt^2 - \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 = 0 \quad \Rightarrow \quad \frac{dr}{dt} = \Big(1 - \frac{2GM}{r}\Big) < 1$$

Time delay

Time runs slower in a gravitational field is compared to flat spacetime. For a stationary observer ($dr = d\theta = d\phi = 0$), the proper time is

- in Schwarzschild: $d\tau = \sqrt{1 - \frac{2GM}{r}}dt$
- in flat spacetime: $d\tau = dt$

where t is the coordinate time measured by an observer at infinity.

Black Holes

The Schwarzschild solution describes static black holes with no electric charge.

- The Schwarzschild radius $R_s = 2GM$ is a coordinate singularity of the geometry: spacetime curvature is finite for $r = R_S$, and this feature is an artifact of the coordinate system we are using, which are not suitable to describe this geometry for $r \leq 2GM$.
- What does it happen to the metric when $r < 2GM$?
  - if $r > 2GM$: $g_{tt} < 0$ and $g_{rr} > 0$

– if $r < 2GM$: $g_{tt} > 0$ and $g_{rr} < 0$

Thus considering a worldline along the t-axes (describing two events happening in the same position at different time):

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 \begin{array}{l} \nearrow \quad < 0 \,\text{for}\, r > 2GM \;\Rightarrow\; \text{TIMELIKE} \\ \searrow \quad > 0 \,\text{for}\, r < 2GM \;\Rightarrow\; \text{SPACELIKE} \end{array}$$

thus passing from $r > 2GM$ to $r < 2GM$ the time and space character of the coordinates do reverse.

- $g_{tt} = -\Big(1 - \frac{2GM}{r}\Big) \xrightarrow{r \to 2GM} 0 \;\Rightarrow\;$ surface of infinite redshift.

Behaviour of light cones along radial paths

Along radial lines $d\theta = d\phi = 0$, and along the path of light rays $ds^2 = 0$, then the slope of the light cone is:

$$\frac{dt}{dr} = \pm\Big(1 - \frac{2GM}{r}\Big)^{-1}$$

($-$ and $+$ refer to outgoing and ingoing light rays)

- $\frac{dt}{dr} \to \pm 1$ for $r \to \infty \Rightarrow$ describes light cones in ordinary Minkowsky spacetime
- Focusing only on the outgoing curves (sign +), $\frac{dt}{dr} \to +\infty$ for $r \to 2GM$ $\Rightarrow$ the light cones become more and more narrow while getting closer to the event horizon.
- Passing from $r > 2GM$ to $r < 2GM$ flips the inner and the outer part of the light cones: the role of r and t is exchanged, and for outgoing particles r decreases as the time flows: the particles are bound to move towards $r = 0$.

Path of a radially infalling particle

We want to describe what happens to an observer falling freely and radially into a spherical black hole from a fixed distance $r_0 > 2GM$. We want to describe his motion:

- from his point of view;
- from the point of view of another observer, sitting still at the same distance r_0.

Let's suppose also that the free-falling observer carries a lamp which emits a sequence of light pulses at constant proper time separation $\Delta\tau$. We see that:

- The proper time that the observer takes to reach a given distance $r < r_0$ is:
$$\tau = -\frac{1}{\sqrt{2GM}}\frac{3}{2}(r^{\frac{3}{2}} - r_0^{\frac{3}{2}})$$
For the infalling observer, nothing special happens at $r = 2GM$. Neglecting the action of tidal forces, he will just cross the horizon and keep on falling until when, after a finite amount of proper time, he will reach the central singularity at $r = 0$.

- The other observer sitting still at r_0 measures the time with the coordinate time t. He sees the freely-falling observer reaching the distance r such that $2GM < r < r_0$ after a time:
$$t = -\frac{2}{3\sqrt{2GM}}\left[r^{\frac{3}{2}} - r_0^{\frac{3}{2}} + 6GM\left(\sqrt{r} - \sqrt{r_0}\right)\right] + 2GM\ln\left[\frac{\left(\sqrt{r}+\sqrt{2GM}\right)\left(\sqrt{r_0}-\sqrt{2GM}\right)}{\left(\sqrt{r_0}+\sqrt{2GM}\right)\left(\sqrt{r}+\sqrt{2GM}\right)}\right]$$
This means that:
  - if both r and r_0 are much bigger than the Schwarzschild radius $2GM$, than $t \approx \tau$, and the light pulses will reach the still observer at separation $\Delta t \approx \Delta\tau$;
  - $t \to \infty$ for $r \to 2GM$. So, the still observer will measure an infinite Schwarzschild coordinate time for the free-falling observer to reach the horizon, and only the light pulses emitted before the falling observer crosses the horizon will reach him. The still observer will receive the light pulses at intervals $\Delta t < \Delta t' < \Delta t" \ldots$ which grow larger and larger, and the pulses of light will be more and more redshifted.

Eddington-Finkelstein coordinates

Rewritten in this set of coordinates the metric is no longer singular at $r = 2GM$. They are defined as the set of coordinates $(\bar{t}, r, \theta, \phi)$ such that the new time coordinate $\bar{t}$ is:
$$\bar{t} = t + 2GM\ln|r - 2GM|$$

The metric becomes:
$$ds^2 = -\left(1 - \frac{2GM}{r}\right)d\bar{t}^2 + \left(1 + \frac{2GM}{r}\right)dr^2 + \frac{4GM}{r}drd\bar{t} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
and it is regular for $r = 2GM$. Studying the behaviour of radial light cones we find that:
$$\frac{d\bar{t}}{dr} = -1 \tag{1a}$$
$$\frac{d\bar{t}}{dr} = \frac{r + 2GM}{r - 2GM} \tag{1b}$$

- (1a) is always oriented at -45 deg with respect to the vertical axis
- (1b) tends to $+1$ (flat space) for $r \to \infty$ and it tends to infinity for $r \to 2GM$, and when $0 < r < 2GM$ it decreases until the limit value -1.
- the future light cone is always directed towards $r \to 0$

Remarks:

- The surface $r = 2GM$ is a coordinate singularity which can be removed via a change of coordinates.
- $r = 2GM$ defines a membrane called *event horizon* of the black hole. It is possible for future directed trajectories to cross it from $r > 2GM$ to $r < 2GM$, but the reverse is not possible.
- Moving towards smaller r, the light cones begin to tip over. At $r = 2GM$ lightlike directions are stationary.
- For $r < 2GM$ all future directed lightlike and timelike trajectories are directed towards $r = 0$.

Cosmology

Cosmology is the study of the universe as a whole. It is based upon the:
Cosmological Principle: The universe is homogeneous and isotropic.

- Homogeneous: the metric is the same throughout space.
- Isotropic: from any point space look the same in any direction.

Remarks.

- The cosmological principle is supported by observation
- Observation shows that distant galaxies are recessing from us: the universe is not static, hence it is homogeneous and isotropic only in space, and not in time. The metric can be put in the form:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x)dx^i dx^j$$

  where

  - t is the cosmological time, $x^i, i = 1, 2, 3$ are the three spatial coordinates, orthogonal to the time direction for an observer sitting at constant x^i(*comoving*).
  - $a(t)$ is the *scale factor*: it says how much big the “space part” of spacetime is at any given coordinate time t.
  - $\gamma_{ij}(x)$ is the *maximally symmetric* metric on the three-dimensional space.

- Implementing the homogeneity and isotropy conditions one finds:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

This metric is called *Robertson-Walker* metric. The parameter K can assume the following values:

  - $K = 1 \Rightarrow$ the spatial part has constant positive curvature, it is closed;
  - $K = 0 \Rightarrow$ the spatial part has zero curvature, it is flat;
  - $K = -1 \Rightarrow$ the spatial part has constant negative curvature, it is open.

- The Einstein equations become a set of differential equations for the scale factor $a(t)$.

Matter and energy content of the universe

The universe is modelled as a perfect fluid at rest in comoving coordinates:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$$

with $U^\mu = (1, 0, 0, 0) \Rightarrow U_\mu = (-1, 0, 0, 0)$. It is convenient to consider $T^\mu{}_\nu = g^{\mu\alpha}T_{\alpha\nu}$

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- The trace of the stress-energy tensor is:

$$T^\mu{}_\mu = -\rho + 3p$$

- The stress-energy tensor is conserved: $D_\mu T^\mu{}_\nu = 0$. The equation for the conservation of energy is obtained selecting $\nu = 0$. One gets:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

where the dot indicates a derivative by the cosmological time t.

- An *equation of state* relates the pressure p and the energy density ρ:

$$p = \omega\rho \qquad \text{where} \quad \omega = \text{constant}$$

- The conservation of energy leads to:

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(\omega + 1) \qquad \Rightarrow \qquad \rho \propto a^{-3(\omega+1)}$$

Examples:

– Dust: collisionless and non-relativistic matter $\Rightarrow$ no pressure: $p = 0$. Then $\omega = 0$ and:

$$\rho_M \propto a^{-3}$$

The number density of particles decreases as the universe expands.

– Radiation: either proper EM radiation or particles moving with velocity close to the speed of light. One has $p_R = \frac{1}{3}\rho_R$ and:

$$\rho_M \propto a^{-4}$$

The energy density of radiation falls faster than the energy density of dust.

– Vacuum: represented by a vacuum stress-energy tensor $T_{\mu\nu}^{(\mathrm{vac})} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}$. Is like a perfect fluid with $\omega = -1$, from which we get:

$$\rho_{(\mathrm{vac})} = constant$$

the vacuum energy density is independent on size and shape of the universe.

- If ρ_M prevails over the others: universe is matter dominated. If ρ_R prevails, universe is radiation dominated. If $\rho_{(\mathrm{vac})}$ prevails: universe is vacuum dominated.

Einstein equations

Neglecting the cosmological constant term, the Einstein equations become a pair of coupled equations determining the evolution of the scale factor:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

These equations are called *Friedmann equations*. The Robertson-Walker metric whose scale factor is a solution of these equations is called *Friedmann-Robertson-Walker* metric.

Cosmological parameters

- Hubble parameter: defines the expansion rate of the universe:

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

- Hubble constant H_0 is the Hubble parameter at present time: 70 km/sec/Mpc $< H_0 <$ 100 km/sec/Mpc.
- Hubble lenght: $d_H = H_0 c^{-1} \approx 3 \times 10^3 h^{-1} Mpc$;

- Hubble time: $t_H = H_0^{-1} = 9.78 \times 10^9 h^{-1} yr.$
- Deceleration parameter: measures the rate of change of the expansion velocity:
$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$
- Density parameter:
$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{\mathrm{C}}}$$
where the critical density $\rho_{\mathrm{C}} = \frac{3H^2}{8\pi G}$ determines the sign of K, *i.e.*, universe's geometry:
  - $\Omega < 1$ ($\Leftrightarrow$ $\rho < \rho_{\mathrm{C}}$) $\Leftrightarrow$ $K < 0$ (open universe);
  - $\Omega = 1$ ($\Leftrightarrow$ $\rho = \rho_{\mathrm{C}}$) $\Leftrightarrow$ $K = 0$ (flat universe);
  - $\Omega > 1$ ($\Leftrightarrow$ $\rho > \rho_{\mathrm{C}}$) $\Leftrightarrow$ $K > 0$ (closed universe);

Hubble Law

- Obtained experimentally, relates the recessing velocity of distant galaxies to their distance:
$$v = H_0 d$$
- Can be obtained directly from the Friedmann-Robertson-Walker metric by computing the radial distance between distant galaxies.

Scale Factor evolution

Obtained by solving the Friedmann-Robertson-Walker equations once they are given the spatial curvature K and the energy densities ρ_R, ρ_M, and $\rho_{(\mathrm{vac})}$.

- If $\rho > 0$ AND $p \geq 0$, and $\Lambda = 0$ (energy or matter dominated universe), we get:
$$\ddot{a} < 0$$
The universe is decelerating, and we know from observations that the universe is expanding. It was expanding faster in the past, and we can trace this expansion back to the singular state at $t = 0$: big bang. In particular
  - if $K \leq 0$ (flat and open cases):
  $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |K|$$
  The right hand side is greater than zero. $\dot{a}$ has never passed through the zero, and it must have been positive all the time: open and flat universes expand forever. But
  $$\dot{a}^2 \xrightarrow{a(t)\to\infty} |K|$$

thus

  - $K = -1$: evolution approaches the limiting value $\dot{a} = 1$;
  - $K = 0$: universe keeps expanding but more and more slowly.

- $K = +1$ (closed universe). We get:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1$$

For $\dot{a}^2$ not to become negative, the universe cannot expand forever, but $a(t)$ must have an upper bound a_{vac}. As a approaches a_{vac}, $\ddot{a}$ is always negative: the universe keeps on contracting. *Closed universes are closed in time too.*

- The precise form of $a(t)$ changes accordingly if the universe is matter or radiation dominated:
  - Radiation dominated: $a(t) \propto \sqrt{t}$.
  - Matter dominated: $a(t) \propto t^{\frac{2}{3}}$.

- Empty universes (only vacuum energy): have either $p < 0$ or $\rho < 0$, and previous considerations don't hold anymore. We can distinguish:
  - $\Lambda < 0$: it exists only an *open* $(K = -1)$ solution with $a(t) \propto e^{Ht}$. This is called anti de Sitter universe.
  - $\Lambda > 0$: they exist solutions for $K = -1$, $K = 0$, and $K = +1$, all called de Sitter universes.