Spacetime and Gravity: Assignment 4 Solutions

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In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

We are given the line element of a two dimensional hyperbolic space:

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) \quad (1)$$

So the metric is:

$$g_{\mu\nu} = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2)$$

and the inverse metric is

$$g^{\mu\nu} = y^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3)$$

We proceed to calculate the Christoffel symbols of this space by using:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\tau}(\partial_\beta g_{\tau\gamma} + \partial_\gamma g_{\tau\beta} - \partial_\tau g_{\beta\gamma}) \quad (4)$$

So the non-vanishing Christoffel symbols are:

$$\begin{aligned}
\Gamma^1_{21} &= \frac{1}{2}g^{11}(\partial_2 g_{11}) && (5)\\
&= \frac{1}{2}y^2\partial_y(\frac{1}{y^2}) && (6)\\
&= -\frac{1}{y} = \Gamma^1_{12} && (7)\\
\Gamma^2_{11} &= -\frac{1}{2}g^{22}\partial_y(\frac{1}{y^2}) && (8)\\
&= \frac{1}{y} && (9)\\
\Gamma^2_{22} &= \frac{1}{2}g^{22}\partial_y(\frac{1}{y^2}) && (10)\\
&= -\frac{1}{y} && (11)
\end{aligned}$$

So what are the geodesic equations for this hyperbolic space? We use the geodesic equation:

$$\frac{d^2x^l}{ds^2} + \Gamma^l_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 0 \tag{12}$$

and expand out the repeated indices

$$\begin{aligned}
0 &= \frac{d^2x^l}{ds^2} + \Gamma^l_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} && (13)\\
0 &= \frac{d^2x^l}{ds^2} + \Gamma^l_{1k}\frac{dx^1}{ds}\frac{dx^k}{ds} + \Gamma^l_{2k}\frac{dx^2}{ds}\frac{dx^k}{ds} && (14)\\
0 &= \frac{d^2x^l}{ds^2} + \Gamma^l_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^l_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} && (15)\\
&\quad + \Gamma^l_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^l_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} && (16)
\end{aligned}$$

As before, whenever we have a free index (in this case the index l) we can manually pick it, so pick $l = 1$ first,

$$\begin{aligned}
\frac{d^2x^1}{ds^2} + \Gamma^1_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^1_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} &= 0 && (17)\\
\Rightarrow \ddot{x} - \frac{1}{y}\dot{x}\dot{y} - \frac{1}{y}\dot{x}\dot{y} &= 0 && (18)\\
\ddot{x} - \frac{2}{y}\dot{x}\dot{y} &= 0 && (19)
\end{aligned}$$

Next, pick $l = 2$

$$\begin{aligned}
\ddot{y} + \Gamma^2_{11}\dot{x}^2 + \Gamma^2_{22}\dot{y}^2 &= 0 && (20)\\
\Rightarrow \ddot{y} + \frac{1}{y}(\dot{x}^2 - \dot{y}^2) &= 0 && (21)
\end{aligned}$$

These two equations are the geodesic equations of our hyperbolic space. We are now set to calculate the Riemann Tensor of the space. We use:

$$R^\epsilon_{\mu\nu\sigma} = -\partial_\sigma\Gamma^\epsilon_{\mu\nu} + \partial_\nu\Gamma^\epsilon_{\mu\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\epsilon_{\alpha\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\epsilon_{\alpha\sigma} \tag{22}$$

Now we can calculate the non vanishing components,

$$\begin{aligned}
R^x_{yxy} &= R^1_{212} && (23)\\
&= -\partial_2\Gamma^1_{21} + \partial_1\Gamma^1_{22} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} && (24)\\
&\quad -\Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22} && (25)\\
&= \partial_y\left(\frac{1}{y}\right) + (-1)^2\frac{1}{y^2} - (-1)^2\frac{1}{y^2} && (26)\\
&= -\frac{1}{y^2} && (27)
\end{aligned}$$

So in order to obtain this in the form R_{xyxy} we must lower the x index with the metric:

$$\begin{aligned} R_{xyxy} &= g_{xx}R^x_{yxy} && (28) \\ &= \frac{1}{y^2} \times -\frac{1}{y^2} && (29) \\ &= -\frac{1}{y^4} = R_{yxyx} = -R_{yxxy} = -R_{xyyx} && (30) \end{aligned}$$

where the last equalities follow from the symmetries of the Riemann tensor. Now we can contract one index on the Riemann Tensor to calculate the Ricci Tensor:

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \tag{31}$$

So again, we expand on the contracted α indices and will have to pick the free μ, ν indices.

$$\begin{aligned} R_{\mu\nu} &= -\partial_\nu\Gamma^\alpha_{\mu\alpha} + \partial_\alpha\Gamma^\alpha_{\mu\nu} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu} && (32) \\ &= -\partial_\nu\Gamma^1_{\mu 1} - \partial_\nu\Gamma^2_{\mu 2} + \partial_1\Gamma^1_{\mu\nu} + \partial_2\Gamma^2_{\mu\nu} && (33) \\ &\quad +\Gamma^1_{\mu\nu}\Gamma^\alpha_{1\alpha} - \Gamma^2_{\mu\nu}\Gamma^\alpha_{2\alpha} - \Gamma^1_{\mu\alpha}\Gamma^\alpha_{1\nu} - \Gamma^2_{\mu\alpha}\Gamma^\alpha_{2\nu} && (34) \\ &= -\partial_\nu\Gamma^1_{\mu 1} - \partial_\nu\Gamma^2_{\mu 2} + \partial_1\Gamma^1_{\mu\nu} + \partial_2\Gamma^2_{\mu\nu} && (35) \\ &\quad +\Gamma^1_{\mu\nu}\Gamma^1_{11} + \Gamma^1_{\mu\nu}\Gamma^2_{12} + \Gamma^2_{\mu\nu}\Gamma^1_{21} + \Gamma^2_{\mu\nu}\Gamma^2_{22} && (36) \\ &\quad -\Gamma^1_{\mu 1}\Gamma^1_{1\nu} - \Gamma^1_{\mu 2}\Gamma^2_{1\nu} - \Gamma^2_{\mu 1}\Gamma^1_{2\nu} - \Gamma^2_{\mu 2}\Gamma^2_{2\nu} && (37) \end{aligned}$$

Now pick $\mu = \nu = 1$,

$$\begin{aligned} R_{11} &= \partial_2\Gamma^2_{11} + \Gamma^2_{12}\Gamma^1_{21} + \Gamma^2_{11}\Gamma^2_{22} && (38) \\ &\quad -\Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{11}\Gamma^1_{21} && (39) \\ &= -\frac{1}{y^2} - \frac{1}{y^2} - \frac{1}{y^2} + \frac{1}{y^2} + \frac{1}{y^2} && (40) \\ &= -\frac{1}{y^2} && (41) \end{aligned}$$

and picking $\mu = \nu = 2$,

$$\begin{aligned} R_{22} &= -\partial_2\Gamma^1_{21} - \partial_2\Gamma^2_{22} + \partial_2\Gamma^2_{22} && (42) \\ &\quad +\Gamma^2_{22}\Gamma^1_{21} + \Gamma^2_{22}\Gamma^2_{22} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{22}\Gamma^2_{22} && (43) \\ &= -\frac{1}{y^2} - \frac{1}{y^2} + \frac{1}{y^2} && (44) \\ &= -\frac{1}{y^2} && (45) \end{aligned}$$

and finally picking $\mu = 1, \nu = 2$ we obtain

$$R_{12} = R_{21} = -\partial_2\Gamma^1_{11} - -\partial_2\Gamma^2_{12} + \partial_1\Gamma^1_{12} + \partial_2\Gamma^2_{12} = 0 \tag{46}$$

So writing out the Ricci tensor in matrix form we see,

$$R_{\mu\nu} = -\frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{47}$$

and hence

$$R_{\mu\nu} = -g_{\mu\nu} \tag{48}$$

as required. This is the consequence of the two-dimensional nature of the space. In two dimensions the cosmological constant vanishes and thus Einstein's field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{49}$$

in a vacuum. Since $R = g^{\mu\nu}R_{\mu\nu} = -2I_2$ then we see that $R_{\mu\nu} = -g_{\mu\nu}$ immediately.

1 Summary of important concepts

1. The geodesic equations are derived from the variation of the action for the motion of a free relativistic particle. They are the equations of motion of a particle which is subjected to a purely gravitational background, i.e. a particle "falling" through a gravitational field follows geodesics of the spacetime created by the gravitational source.
2. Remember the important symmetries of the Riemann tensor

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha} \tag{50}$$