Spacetime and Gravity: Assignment 1 Solutions

In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

Given the Lorentz transformations,

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}} \tag{1}$$

We show,

$$\begin{aligned}
-t'^2 + x'^2 &= -\frac{(t^2 - 2vxt + v^2x^2)}{1 - v^2} + \frac{x^2 - 2vtx + v^2t^2}{1 - v^2} && (2)\\
&= \frac{x^2 - t^2 - v^2x^2 + v^2t^2}{1 - v^2} && (3)\\
&= \frac{-t^2(1 - v^2) + x^2(1 - v^2)}{1 - v^2} && (4)\\
&= -t^2 + x^2 && (5)
\end{aligned}$$

We set $v = \tanh(\theta)$ and make use of the following identity:

$$\frac{1}{\sqrt{1 - \tanh^2(\theta)}} = \cosh(\theta) \tag{6}$$

Then,

$$\begin{aligned}
t' &= \frac{t - \tanh(\theta)x}{\sqrt{1 - \tanh^2(\theta)}} && (7)\\
&= (t - \tanh(\theta)x)\cosh(\theta) && (8)\\
&= t\cosh(\theta) - x\sinh(\theta) && (9)
\end{aligned}$$

And,

$$\begin{aligned}
x' &= \frac{x - \tanh(\theta)t}{\sqrt{1 - \tanh^2(\theta)}} && (10)\\
&= (x - \tanh(\theta)t)\cosh(\theta) && (11)\\
&= x\cosh(\theta) - t\sinh(\theta) && (12)
\end{aligned}$$

We have derived the Lorentz transformations in terms of the newly introduced θ variable. We proceed to verify the invariance of the interval:

$$
\begin{aligned}
-t'^2 &= -t^2\cosh^2(\theta) - x^2\sinh^2(\theta) + 2xt\cosh(\theta)\sinh(\theta) && (13)\\
x'^2 &= x^2\cosh^2(\theta) + t^2\sinh^2(\theta) - 2xt\cosh(\theta)\sinh(\theta) && (14)
\end{aligned}
$$

and hence

$$
\begin{aligned}
-t'^2 + x'^2 &= -t^2(\cosh^2(\theta) - \sinh^2(\theta)) + x^2(\cosh^2(\theta) - \sinh^2(\theta)) && (15)\\
&= -t^2 + x^2. && (16)
\end{aligned}
$$

We have seen an example of a very important general principle, the interval is invariant under Lorentz transformations.

The matrix form of a Lorentz transformation with factors of c restored and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ reads

$$
\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v\gamma}{c} \\ -\frac{v\gamma}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \qquad (17)
$$

To see the effect of two successive Lorentz transformations, multiply two such matrices together

$$
\begin{aligned}
\gamma \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} &= \gamma_2 \begin{pmatrix} 1 & -\frac{v_2}{c} \\ -\frac{v_2}{c} & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & -\frac{v_1}{c} \\ -\frac{v_1}{c} & 1 \end{pmatrix} && (18)\\
&= \gamma_1\gamma_2 \begin{pmatrix} 1 + \frac{v_1 v_2}{c^2} & -\frac{v_1+v_2}{c} \\ -\frac{v_1+v_2}{c} & 1 + \frac{v_1 v_2}{c^2} \end{pmatrix} && (19)\\
&= \gamma_1\gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right) \begin{pmatrix} 1 & -\frac{\frac{v_1+v_2}{c}}{1+\frac{v_1+v_2}{c^2}} \\ -\frac{\frac{v_1+v_2}{c}}{1+\frac{v_1+v_2}{c^2}} & 1 \end{pmatrix} && (20)
\end{aligned}
$$

which is the same as a single transformation with $v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$ since

$$
\begin{aligned}
\gamma_1\gamma_2 \left(1 + \frac{v_1 v_2}{c^2}\right) &= \left[\left(1 - \frac{v_1^2}{c^2}\right)\left(1 - \frac{v_2^2}{c^2}\right)\left(1 + \frac{v_1 v_2}{c^2}\right)^{-2}\right]^{-1/2} && (21)\\
&= \left[\left(1 - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2} + \frac{v_1^2 v_2^2}{c^4}\right)\left(1 + \frac{v_1 v_2}{c^2}\right)^{-2}\right]^{-1/2} && (22)
\end{aligned}
$$

is the same as

$$
\begin{aligned}
\gamma = \left[1 - \frac{v^2}{c^2}\right]^{-1/2} &= \left[1 - \left(\frac{\frac{v_1+v_2}{c}}{1 + \frac{v_1 v_2}{c^2}}\right)^2\right]^{-1/2} && (23)\\
= \left[\left(1 + 2\frac{v_1 v_2}{c^2} + \frac{v_1^2 v_2^2}{c^4} - \frac{v_1^2}{c^2} - 2\frac{v_1^2 v_2^2}{c^4} - \frac{v_2^2}{c^2}\right)\left(1 + \frac{v_1 v_2}{c^2}\right)^{-2}\right]^{-1/2} & && (24)\\
= \left[\left(1 - \frac{v_1^2}{c^2} - \frac{v_2^2}{c^2} + \frac{v_1^2 v_2^2}{c^4}\right)\left(1 + \frac{v_1 v_2}{c^2}\right)^{-2}\right]^{-1/2}. & && (25)
\end{aligned}
$$

There is another way of seeing this. If we put the equations

$$t' = t\cosh(\zeta) - x\sinh(\zeta) \qquad\qquad x' = x\cosh(\zeta) - t\sinh(\zeta) \tag{26}$$

into matrix form then we have

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\zeta) & -\sinh(\zeta) \\ -\sinh(\zeta) & \cosh(\zeta) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{27}$$

This matrix equation allows us to express our primed co-ordinates in terms of our unprimed co-ordinates. Let's now perform a second transformation so our double-primed co-ordinates are given by

$$\begin{pmatrix} ct'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh(\zeta_2) & -\sinh(\zeta_2) \\ -\sinh(\zeta_2) & \cosh(\zeta_2) \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{28}$$

$$= \begin{pmatrix} \cosh(\zeta_2) & -\sinh(\zeta_2) \\ -\sinh(\zeta_2) & \cosh(\zeta_2) \end{pmatrix} \begin{pmatrix} \cosh(\zeta_1) & -\sinh(\zeta_1) \\ -\sinh(\zeta_1) & \cosh(\zeta_1) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} \cosh(\zeta_2)\cosh(\zeta_1) + \sinh(\zeta_2)\sinh(\zeta_1) & -\cosh(\zeta_2)\sinh(\zeta_1) - \sinh(\zeta_2)\cosh(\zeta_1) \\ -\cosh(\zeta_2)\sinh(\zeta_1) - \sinh(\zeta_2)\cosh(\zeta_1) & \cosh(\zeta_2)\cosh(\zeta_1) + \sinh(\zeta_2)\sinh(\zeta_1) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} \cosh(\zeta_1 + \zeta_2) & -\sinh(\zeta_1 + \zeta_2) \\ -\sinh(\zeta_1 + \zeta_2) & \cosh(\zeta_1 + \zeta_2) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{31}$$

We can clearly see that this takes the form of equation (27) with $\zeta = \zeta_1 + \zeta_2$. This means that it is in fact a Lorentz Transformation. The value of v for this combined Lorentz Transformation is

$$v = c\tanh(\zeta) = c\tanh(\zeta_1 + \zeta_2) = c\left(\frac{\tanh(\zeta_1) + \tanh(\zeta_2)}{1 + \tanh(\zeta_1)\tanh(\zeta_2)}\right) \tag{32}$$

$$= c\left(\frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1 v_2}{c^2}}\right) = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \tag{33}$$

If we set $v_1 = c$ then $v = \frac{c+v_2}{1+\frac{cv_2}{c^2}} = \frac{c(c+v_2)}{c+v_2} = c$, i.e. the combined velocity is still c, no matter what v_2 is. This relativistic velocity addition shows that nothing can exceed the speed of light.

2.

We are given the energy and momentum Lorentz transformations:

$$E' = \frac{E - vp}{\sqrt{1 - v^2}} \qquad p' = \frac{p - vE}{\sqrt{1 - v^2}} \tag{34}$$

Consider

$$-E'^2 + p'^2 = \frac{-E^2 - v^2p^2 + 2vpE}{1 - v^2} + \frac{p^2 - v^2E^2 - 2pvE}{1 - v^2} \tag{35}$$

$$= \frac{-E^2(1 - v^2) + p^2(1 - v^2)}{1 - v^2} \tag{36}$$

$$= -E^2 + p^2 \tag{37}$$

and define $-E^2 + p^2 = -m^2$. We boost to a frame in which $p = 0$ and use dimensional analysis to re-introduce the missing factors of c. We denote with $[x]$ the units of the variable x.

We have,

$$E^2 = m^2 \tag{38}$$

and in SI units,

$$[E]^2 = [m]^2[c]^x \tag{39}$$

Where x is an unknown power. Writing out the expression explicitly one obtains:

$$\begin{aligned}
(Kgm^2s^{-2})^2 &= (Kg)^2(ms^{-1})^x && (40)\\
m^4s^{-4} &= (ms^{-1})^x && (41)
\end{aligned}$$

¿From which we obtain $x = 4$. We substitute back into our original expression,

$$\begin{aligned}
E^2 &= m^2c^4 && (42)\\
E &= mc^2 && (43)
\end{aligned}$$

and we have derived Einstein's famous energy equation. N.B. $E = -mc^2$ is also a possible solution but it remains non-physical outside of a quantum mechanical context.

3.

We are given the line element:

$$\begin{aligned}
ds^2 &= g_{\mu\nu}dx^\mu dx^\nu && (44)\\
&= -f(y)dt^2 + 2f(y)\gamma dxdt + f(y)dx^2 + dy^2 + dz^2 && (45)
\end{aligned}$$

from which we can extract the metric:

$$g_{\mu\nu} = \begin{pmatrix} -f(y) & f(y)\gamma & 0 & 0 \\ f(y)\gamma & f(y) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{46}$$

Now $g^{\mu\nu}$ is defined to be the inverse of the metric. So we have to invert the matrix we previously obtained. We denote $f(y) = f$ in the following. Because $g_{\mu\nu}$ is in block-diagonal form we can invert it block by block. Recall for a general matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{47}$$

Therefore

$$\begin{pmatrix} -f & f\gamma \\ f\gamma & f \end{pmatrix}^{-1} = \frac{-1}{f^2+f^2\gamma^2}\begin{pmatrix} f & -f\gamma \\ -f\gamma & -f \end{pmatrix} \tag{48}$$

$$= \frac{-1}{f(1+\gamma^2)}\begin{pmatrix} 1 & -\gamma \\ -\gamma & -1 \end{pmatrix} \tag{49}$$

And

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{50}$$

so that

$$g^{\mu\nu} = \frac{1}{f(1+\gamma^2)}\begin{pmatrix} -1 & \gamma & 0 & 0 \\ \gamma & 1 & 0 & 0 \\ 0 & 0 & f(1+\gamma^2) & 0 \\ 0 & 0 & 0 & f(1+\gamma^2) \end{pmatrix} \tag{51}$$

We perform the coordinate transformation

$$T = \sqrt{1+\gamma^2}t \qquad X = x + \gamma t \tag{52}$$

and proceed to calculate $-dT^2 + dX^2$.

$$-dT^2 + dX^2 = -(1+\gamma^2)dt^2 + dx^2 + \gamma^2 dt^2 + 2\gamma dxdt \tag{53}$$

$$= -dt^2 + 2\gamma dxdt + dx^2 \tag{54}$$

This means that we can express the old line element as

$$ds^2 = -f(y)dT^2 + f(y)dX^2 + dy^2 + dz^2 \tag{55}$$

and we see that the metric in these new coordinates is diagonal:

$$g'_{\mu\nu} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{56}$$

However we have just calculated that $-dT^2 + dX^2 \neq -dt^2 + dx^2$. The interval isn't preserved so this is not a Lorentz transformation! (See Qu.1)

4.

The line element is

$$ds^2 = -f(t,x)dt^2 + 2g(t,x)dtdx + h(t,x)dx^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{57}$$

We can proceed to extract the metric, denoting $f(t,x) = f$,$g(t,x) = g$ and $h(t,x) = h$

$$g_{\mu\nu} = \begin{pmatrix} -f & g \\ g & h \end{pmatrix} \tag{58}$$

Then the inverse metric is

$$g^{\mu\nu} = \frac{1}{fh + g^2} \begin{pmatrix} -h & g \\ g & f \end{pmatrix} \tag{59}$$

To write out the full expressions of the given contractions we make repeated use of the summation convention.

$$\begin{aligned}
x^\mu x_\mu &= g_{\nu\mu} x^\nu x^\mu && (60)\\
&= g_{1\mu} x^1 x^\mu + g_{2\mu} x^2 x^\mu && (61)\\
&= g_{11} x^1 x^1 + g_{22} x^2 x^2 + g_{12} x^1 x^2 + g_{21} x^2 x^1 && (62)\\
&= -ft^2 + hx^2 + 2gxt && (63)
\end{aligned}$$

Similarly,

$$\begin{aligned}
E_\mu &= g_{\mu\nu} E^\nu && (64)\\
&= g_{\mu 1} E^1 + g_{\mu 2} E^2 && (65)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E_1 &= g_{11} E^1 + g_{12} E^2 && (66)\\
&= -fE + gp && (67)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E_2 &= g_{21} E^1 + g_{22} E^2 && (68)\\
&= hp + gE && (69)
\end{aligned}$$

Therefore

$$E_\mu = (-fE + gp, hp + gE) \tag{70}$$

The method for all such contractions is the same,

$$\begin{aligned}
E_\mu x^\mu &= g_{1\nu} E^\nu x^1 + g_{2\nu} E^\nu x^2 && (71)\\
&= g_{11} E^1 x^1 + g_{12} E^1 x^2 + g_{21} E^2 x^1 + g_{22} E^2 x^2 && (72)\\
&= -fEt + gpt + gEx + hpx && (73)\\
&= -fEt + g(pt + Ex) + hpx && (74)
\end{aligned}$$

and

$$\begin{aligned}
E^\nu E_\nu &= g_{\mu\nu} E^\mu E^\nu && (75)\\
&= g_{1\nu} E^1 E^\nu + g_{2\nu} E^2 E^\nu && (76)\\
&= g_{11} E^1 E^1 + g_{12} E^1 E^2 + g_{21} E^2 E^1 + g_{22} E^2 E^2 && (77)\\
&= -fE^2 + 2gpE + hp^2. && (78)
\end{aligned}$$

Finally,

$$
\begin{aligned}
x^\mu E_\nu x^\nu E_\mu &= x^\mu E_\mu x^\nu E_\nu && (79)\\
&= (-fEt + g(pt + Ex) + hpx)^2 && (80)
\end{aligned}
$$

and

$$
\begin{aligned}
x^\mu E^\nu x_\mu E_\nu &= x^\mu x_\mu E^\nu E_\nu && (81)\\
&= (-ft^2 + hx^2 + 2gtx)(-fE^2 + 2gpE + hp^2) && (82)
\end{aligned}
$$

Note that for these last two contractions we could of explicitly carried out the summations using the metric components but it was much simpler to rearrange the expressions into forms which we can recognize by using the commutation properties of E^μ and x^μ.

1 Summary of Important Concepts

The interval is invariant under Lorentz transformations and hence only those transformations which leave it invariant are in fact Lorentz.

The summation convention states that:

$$x^\mu x_\mu = \sum_{\mu=0}^{d} x^\mu x_\mu \tag{83}$$

where d is the dimension of the system in consideration.

Using the summation convention one can extract the metric out of any line element.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{84}$$

Make sure to rearrange contractions into forms that are easily recognized by making use of commutation relations of variables.