QUANTUM PHYSICS 2013

A1. $E = \frac{hc}{\lambda} = \frac{2\pi \hbar c}{\lambda} = \frac{6.28 \times 197 \text{ MeV fermi}}{10^{-9} m} =$

$\simeq 1.23$ keV $\boxed{E \simeq 1.23 \text{ keV}}$

A2. Wien's law states that $\boxed{\lambda_{MAX} T = \text{const}}$

where λ_{MAX} is the wavelength where the emissive power $R(\lambda, T)$ has a maximum. T is the temperature of the black body.

A3. Planck assumed that the energy of the oscillators is quantised (as opposed to continuous) i.e.

$\boxed{E = n \varepsilon_0}$ $n = 0, 1, 2, \ldots$ for a fixed quantum of energy ε_0.

A4. In order to have photoelectric emission we need $E_{photon} > W$.

$E_{photon} = \frac{hc}{\lambda} = \frac{197 \cdot 2\pi \cdot \cancel{10^{-9}}}{5 \cancel{10^{-9}}}$ eV ~ 247.5 eV

Since $W \simeq 4.08$ eV and

$\boxed{E_{photon} > W}$ there will be photoelectric effect.

A5. $\lambda_{\text{de Broglie}} = \frac{h}{p}$. Since $E \gg m_e c^2$

we have $E \simeq pc \Rightarrow pc \simeq 10$ GeV.

Then $\lambda_{\text{de Broglie}} = \frac{hc}{pc} \sim \frac{2\pi \cdot 197 \,\cancel{\text{MeV}}\,\text{fermi}}{10 \cdot 10^3 \,\cancel{\text{MeV}}}$

$\simeq 0.12$ fermi $\Rightarrow \boxed{\lambda_{\text{de Broglie}} \simeq 0.12 \text{ fermi}}$

The electrons are ultra-relativistic since $E \gg mc^2 \Rightarrow$ non-relativistic approximation is inappropriate.

A6. λ must be comparable to a, i.e.

$$\boxed{\lambda \sim a}$$

A7. $$\boxed{\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t) = i\hbar\frac{\partial \psi}{\partial t}(x,t)}$$

A8. $$\boxed{\int_{-a}^{a} dx\, |\psi(x)|^2 = 1}$$

A9. $$\boxed{\langle x \rangle_\psi = \int_{-a}^{a} dx\, x\, |\psi(x)|^2}$$

A 10 - We have to find the solution to

$$\frac{d}{dx} |\psi(x)|^2 = 0$$ or, since ψ is real, $\psi'(x) = 0 \Rightarrow$

$$2x e^{-x} - x^2 e^{-x} = 0 \Rightarrow \boxed{x = 2}$$

($x=0$ is a minimum).

PROBLEM 1

(i) Balancing centripetal and electrostatic effects we get

$$\frac{m v^2}{\cancel{R}} = \frac{Z e^2}{4\pi \varepsilon_0 R^{\cancel{2}}} \quad \text{or}$$

$$\boxed{v^2 = \frac{Z e^2}{4\pi \varepsilon_0 m} \frac{1}{R}}$$

($Z = 1$ for this question)

(ii) $$E = T + V = \frac{1}{2} m v^2 - \frac{Z e^2}{4\pi \varepsilon_0 R} =$$

$$= \frac{1}{2} \cancel{m} \frac{Z e^2}{4\pi \varepsilon_0 \cancel{m} R} - \frac{Z e^2}{4\pi \varepsilon_0 R} \Rightarrow$$

$$\boxed{E = -\frac{1}{2} \frac{Z e^2}{4\pi \varepsilon_0 R}}$$

$$\boxed{|L| = m v R = \left(\frac{Z e^2}{4\pi \varepsilon_0} m R \right)^{\frac{1}{2}}}$$

(iii) Bohr quantisation condition is

$L = n\hbar$ with $n =$ integer $\Rightarrow$

$\boxed{m v R = n\hbar}$

We then have the 2 equations

$$\begin{cases} v^2 R = \dfrac{Z e^2}{4\pi\varepsilon_0 m} \\ v R = \dfrac{n\hbar}{m} \end{cases} \Rightarrow \frac{v^2R^2}{v^2R} = R = \frac{n^2\hbar^2}{m^{\cancel{2}}} \frac{4\pi\varepsilon_0 \cancel{m}}{Z e^2}$$

$$\Rightarrow \boxed{R_n = \hbar^2 \frac{4\pi\varepsilon_0}{m e^2} \frac{n^2}{Z}}$$

Setting $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} \Rightarrow$

$$\boxed{R_n = \underbrace{\left(\frac{\hbar c}{\alpha} \frac{1}{mc^2}\right)}_{\text{Bohr radius}} \frac{n^2}{Z} = a_{Bohr} \frac{n^2}{Z}}$$

For v_n: $\dfrac{v^2R}{vR} = v = \dfrac{Z e^2}{4\pi\varepsilon_0 \cancel{m}} \dfrac{\cancel{m}}{n\hbar} = \dfrac{Z}{n} \alpha c \Rightarrow$

$$\boxed{v_n = \alpha c \frac{Z}{n}}$$

(iv) The energy levels can be found from our earlier formula $E = -\frac{1}{2}\frac{Ze^2}{4\pi\varepsilon_0 R}$ by plugging $R \to R_n$ $\Rightarrow$

$$E_n = -\frac{1}{2}\frac{Ze^2}{4\pi\varepsilon_0 R_n} = -\frac{1}{2}\frac{Ze^2}{4\pi\varepsilon_0}\frac{mZe^2}{4\pi\varepsilon_0\hbar^2}\frac{1}{n^2} =$$

$$= -\frac{1}{2}\frac{Z^2}{n^2}\cdot\frac{mc^2 e^2}{(4\pi\varepsilon_0\hbar c)^2} = -\frac{1}{2}\frac{Z}{n^2}\alpha^2(mc^2)$$

$$\boxed{E_n = -\frac{mc^2}{2}\frac{Z^2}{n^2}\alpha^2} \quad \text{or} \quad E_n \simeq -\frac{Z^2}{n^2}\cdot 13.6\ eV$$

For our Li^{++} ion, $Z=3$ and

$$E_{n=1}^{Li^{++}} = E_{\substack{\text{ground}\\ \text{state}}} = -\frac{9}{1}\cdot(13.6)\ eV \approx -122.4\ eV$$

$$\Rightarrow \boxed{E_{n=1}^{Li^{++}} \simeq -122.4\ eV}$$

Finally, $r_n^{Z} = \frac{r_n^{Z=1}}{Z}$ $\Rightarrow$

$$\boxed{\frac{r_n^{Z=3}}{r_n^{Z=1}} = \frac{1}{3}}$$

PROBLEM 2

(i) $\lambda = \frac{h}{p}$ For a photon $E = cp \Rightarrow$

$$\lambda = \frac{hc}{pc} = \frac{hc}{E} \Rightarrow \boxed{E = \frac{hc}{\lambda}}$$

Numerically : recall that $\hbar c \simeq 197$ MeV fermi
$\Rightarrow$

$$E \approx \frac{6.28 \times 197 \text{ MeV } 10^{-15} \text{ m}}{0.03 \quad 10^{-9} \text{ m}} \sim 41.2 \text{ keV}$$

Energy conservation :

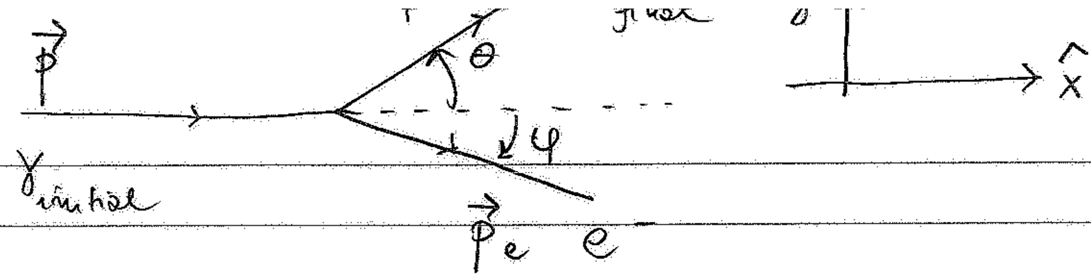
$$E + m_e c^2 = E' + E_e$$

- $E\ (E')$ = energy of the initial (final) photon
- E_e = energy of the final electron = $= m_e c^2 \gamma$

Hence $E' - E = m_e c^2 - E_e =$

$$= m_e c^2 (1-\gamma) < 0 \qquad \text{since } \gamma > 1$$

(ii)



For $\theta = \frac{\pi}{2}$ $\quad \lambda' - \lambda = \frac{h}{mc}\left(1 - \cancel{\cos\left(\frac{\pi}{2}\right)}\right) = \frac{h}{mc} =$

$$= \frac{6.28 \cdot 197 \text{ MeV fermi}}{0.5 \text{ MeV}} \simeq 0.0024 \text{ nm} \equiv \lambda_c$$

($\lambda_c \equiv \frac{h}{mc}$ is the Compton wavelength).

the kinetic energy of the recoiling electron:

from $\quad mc^2 + E = E' + E_e \quad$ we get

$E_e - mc^2 =$ kinetic energy of final electron $=$

$$= E - E' = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = hc\,\frac{\lambda' - \lambda}{\lambda\lambda'}$$

We found, in our case, that $\lambda' = \lambda + \lambda_c \Rightarrow$

$$E_e - mc^2 = hc\,\frac{\lambda_c}{\lambda\lambda'} = hc\,\frac{\lambda_c}{\lambda(\lambda+\lambda_c)} =$$

$$= 6.28 \cdot 197 \text{ MeV } 10^{-15} \cancel{\text{m}} \; \frac{0.0024 \; \cancel{\text{nm}}}{0.03\,(0.03 + 0.0024)\; \cancel{\text{nm}}\; 10^{-9} \cancel{\text{m}}}$$

~ 3.05 keV

$$\Rightarrow \boxed{T_e \sim 3.05 \text{ keV}}$$

(iii) $\vec{p} = \vec{p}\,' + \vec{p}_e$

momentum in initial photon — $\vec{p}$; final photon — $\vec{p}\,'$; final electron — $\vec{p}_e$

or $$\boxed{\vec{p}_e = \vec{p} - \vec{p}\,'}$$

Project this along axes:

- $p_{ex} \equiv p_e \cos\varphi = p - p'\cos\theta = \frac{1}{c}(E - E'\cos\theta)$

since $E = pc$, $E' = p'c$

- $p_{ey} = -p_e \sin\varphi = 0 - p'\sin\theta = -\frac{E'}{c}\sin\theta$

Dividing we get

$$\frac{p_{ey}}{p_{ex}} = -\tan\varphi = -\frac{E'\sin\theta}{E - E'\cos\theta} \Rightarrow$$

$$\boxed{\tan\varphi = \frac{E'\sin\theta}{E - E'\cos\theta}}$$

(IV) Because $\vec{p} \parallel \hat{x}$ we have $p_y' + p_{ey} = 0$ or

$p_{ey} = -p_y' = -|\vec{p}\,'|\sin\theta = -\frac{E'}{c}\sin\theta$

$\boxed{p_{ey} = -\frac{E'}{c}\sin\theta}$. From $\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$

we get, since $E = \frac{hc}{\lambda}$, $E' = \frac{hc}{\lambda'}$,

$$\frac{1}{E'} - \frac{1}{E} = \frac{1-\cos\theta}{mc^2} \Rightarrow \boxed{\frac{1}{E'} = \frac{mc^2 + E(1-\cos\theta)}{E\,mc^2}}$$

Hence $p_{ey} = -\sin\theta \dfrac{E\,mc}{mc^2 + E(1-\cos\theta)}$ or

$$\boxed{p_{ey} = -\sin\theta \, \frac{E}{c} \, \frac{1}{1 + \frac{E}{mc^2}(1-\cos\theta)}}$$

• Note: the sign in p_{ey} depends on the direction we choose for $\hat{y}$. Answer with + is alright!

PROBLEM 3

(i) The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

. The boundary conditions are $\psi(0) = \psi(L) = 0$

(ii) $\psi(x) = A\left[\sin\frac{2\pi x}{L} + 4\cdot\sin\frac{6\pi x}{L}\right]$

Using $\psi^{(n)}(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$, $n = 1, 2, \ldots$

we can rewrite

$$\boxed{\psi(x) = A\sqrt{\frac{L}{2}}\left(\psi^{(2)} + 4\cdot\psi^{(6)}\right)}$$

We determine A by requiring that

$$\int_0^L dx\,|\psi(x)|^2 = 1$$

or

$$|A|^2\frac{L}{2}\left[\int_0^L dx\,|\psi^{(2)}|^2 + 16\int_0^L dx\,|\psi^{(6)}|^2 + 8\int_0^L dx\,\psi^{(2)}\psi^{(6)}\right]$$

$= 1$

Since $\int_0^L dx\,\psi^{(n)*}\psi^{(m)} = \delta_{nm}$, we get

$$|A|^2 \frac{L}{2}(1+16) = 1 \quad \Rightarrow$$

$$|A|^2 = \frac{2}{17L}$$ or, up to an irrelevant phase,

$$\boxed{A = \sqrt{\frac{2}{17L}}}$$

The normalised wave function is therefore

$$\boxed{\psi(x) = \frac{1}{\sqrt{17}}\left(\psi^{(2)} + 4\psi^{(6)}\right)}$$

(iii) $\psi = C_2 \psi^{(2)} + C_6 \psi^{(6)}$ with

$$C_2 = \frac{1}{\sqrt{17}} \qquad C_6 = \frac{4}{\sqrt{17}}$$ - then

$$\langle E \rangle_\psi = E_2 |C_2|^2 + E_6 |C_6|^2 \text{ -}$$

Since $E_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2$, $n = 1, 2 \ldots$ we get

$$\langle E \rangle_\psi = \frac{\hbar^2 \pi^2}{2mL^2}\left[\overset{\nearrow 2\cdot 2}{4} \cdot \frac{1}{17} + \overset{\nearrow 6\cdot 6}{36} \cdot \frac{16}{17}\right] =$$

$$= \frac{\hbar^2 \pi^2}{2mL^2} \frac{580}{17} \quad \Rightarrow$$

$$\boxed{\langle E \rangle_\psi = \frac{290}{17} \frac{\hbar^2 \pi^2}{mL^2}}$$

(IV) After the measurement the energy is

$$\frac{18\pi^2\hbar^2}{mL^2} = E_{n=6} \Rightarrow$$

$$\boxed{\psi(x,t) = \psi^{(6)}(x)\exp\left(-\frac{i E_6 t}{\hbar}\right)}$$

with $\psi^{(6)}(x) = \sqrt{\frac{2}{L}}\sin\frac{6\pi x}{L}$ and

$$E_6 = \frac{18\pi^2\hbar^2}{mL^2}$$

As for $\langle x\rangle$:

$$\langle x\rangle_\psi = \int_0^L dx\ x\,|\psi(x,t)|^2 = \int_0^L dx\ x\,|\psi^{(6)}(x)|^2 =$$

$$= \frac{2}{L}\int_0^L dx\ x\sin^2\left(\frac{6\pi x}{L}\right) = \frac{2}{L}\left(\frac{L}{6\pi}\right)^2\int_0^{6\pi} dz\, z\sin^2 z =$$

$$= \frac{2}{\cancel{L}}\frac{L^{\cancel{2}}}{\cancel{(6\pi)^2}}\frac{\cancel{(6\pi)^2}}{4} = \frac{L}{2} \Rightarrow$$

$$\boxed{\langle x\rangle = \frac{L}{2}}$$

PROBLEM 4

(i) The boundary conditions $\psi(1) = \psi(-1) = 0$ must be obeyed $\Rightarrow$ since $\psi(x) = a(x-b)\sqrt{x+1}$ we need

$\boxed{b = 1}$

(ii) We have $\psi(x) = a(x-1)\sqrt{x+1}$ and now require that $\int_{-1}^{1} dx\, |\psi|^2(x) = 1$

$\Rightarrow$

$$a^2 \int_{-1}^{1} dx\, (x-1)^2 (x+1) = 1 \quad \Rightarrow$$

Since $\int_{-1}^{1} dx\, (x-1)^2 (x+1) = \frac{4}{3}$ we get, up to an irrelevant phase,

$$\boxed{a = \sqrt{\frac{3}{4}}} \quad \Rightarrow \quad \boxed{\psi(x) = \sqrt{\frac{3}{4}}\,(x-1)\sqrt{x+1}}$$

(iii) We have

- $\langle x\rangle_\psi = \int_{-1}^{1} dx \; x \; |\psi(x)|^2 =$

$$= \frac{3}{4}\int_{-1}^{1} dx \; x\,(x-1)^2(x+1)$$

Since $\int_{-1}^{1} dx \; x(x-1)^2(x+1) = -\frac{4}{15}$ we get

$$\langle x\rangle_\psi = -\frac{\cancel{3}}{\cancel{4}}\cdot\frac{\cancel{4}}{\cancel{15}_{5}} = -\frac{1}{5} \Rightarrow \boxed{\langle x\rangle_\psi = -\frac{1}{5}}$$

- $\psi(x)$ is real hence $\langle p\rangle_\psi = 0$ – Explicitly

$$\langle p\rangle_\psi = \int_{-1}^{1} dx \; \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) = \text{since } \psi^* = \psi$$

$$= -\frac{i\hbar}{2}\int_{-1}^{1} dx \frac{d}{dx}[\psi(x)]^2 = 0 \text{ since } \psi(\pm 1) = 0.$$

- The most probable value for x is determined by imposing

$\frac{d}{dx}|\psi(x)|^2 = 0$. In our case we have

$$\frac{d}{dx}|\psi(x)|^2 = 2\psi(x)\frac{d}{dx}\psi(x) = 0$$

Hence we have to calculate $\frac{d}{dx}\left[(x-1)\sqrt{x+1}\right]$:

$$\frac{d}{dx}\left[(x-1)\sqrt{x+1}\right] = \sqrt{x+1} + \frac{x-1}{2\sqrt{x+1}} = \frac{3x+1}{2\sqrt{x+1}}$$

$\Rightarrow$ $\boxed{x = -\frac{1}{3}}$ is the most probable value of x.

(iv) Answer is no since ψ is not a stationary state. We can explicitly check this by computing $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi$ and observing that it is not proportional to ψ.

We already have $\frac{d}{dx}\psi = \left(\frac{3x+1}{2\sqrt{1+x}}\right)\cdot\sqrt{\frac{3}{4}}$

Further differentiating we get

$$\frac{d^2}{dx^2}\psi = \sqrt{\frac{3}{4}}\frac{1}{2\sqrt{1+x}}\left[3 + \frac{(3x+1)\left(-\frac{1}{2}\right)}{1+x}\right] =$$

$$= \sqrt{\frac{3}{4}}\cdot\frac{1}{4\sqrt{1+x}}\left[\frac{6+6x-3x-1}{1+x}\right] = \frac{5+3x}{4(1+x)^{3/2}}\cdot\sqrt{\frac{3}{4}}$$

Hence $\boxed{\frac{d^2}{dx^2}\psi = \frac{5+3x}{(1+x)^{3/2}}\cdot\frac{1}{4}\sqrt{\frac{3}{4}}}$ which is not of the form $E\psi$ for some E $\Rightarrow$ ψ is not a stationary state.