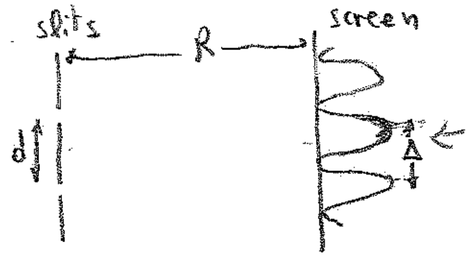
QP 2011

A1 Wiens law states that the peak wavelength in the blackbody spectrum λ_{max} is related to the body temperature T

$$\lambda_{max} T \sim 2.9 \; 10^{-3} \, mK \qquad T \sim \frac{2.9 \; 10^{-3}}{1.06 \; 10^{-3}} K \sim 2.73 \, K$$

A2



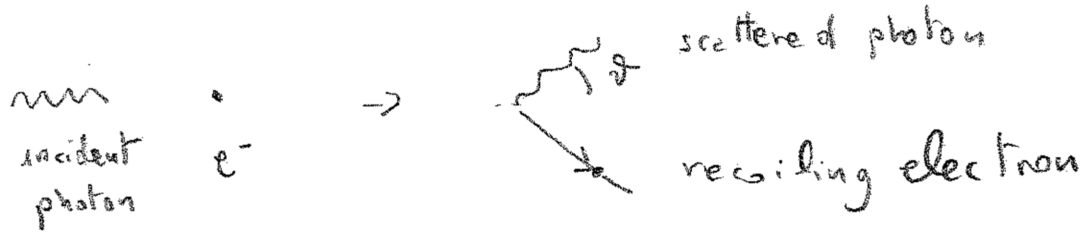
Interference pattern

$$\lambda = \frac{d \, \Delta}{R}$$

A3 If ϕ is the metal's work function and λ_t the threshold frequency, then $\frac{hc}{\lambda_t} = \phi$. The maximum KE when $\lambda = 3 . 10^{-7} m$ is $KE = \frac{hc}{\lambda} - \frac{hc}{\lambda_t} = \frac{hc}{\lambda \lambda_t}(\lambda_t - \lambda)$

$$K = 4.14 \; 10^{-15} eVs \cdot 3 \; 10^{8} \frac{m}{s} \; \frac{5-3}{15 \; 10^{-7} m} \simeq 1.66 \; eV$$

A4



Momentum conservation gives

$$(P_e)_y = \frac{h}{\lambda'} \sin \vartheta \qquad (P_e)_x = \frac{h}{\lambda} - \frac{h}{\lambda'} \cos \vartheta$$

Thus $P_e^2 = (P_e)_y^2 + (P_e^2)_x = \left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2 - 2 \frac{h^2}{\lambda \lambda'} \cos \vartheta$

A5 The KE gained by the electrons thank to the potential V is $KE = |eV|$. The minimal wavelength these electron can produce is λ_{min}

$$|eV| = \frac{hc}{\lambda_{min}} \Rightarrow V = \left|\frac{hc}{e\lambda_{min}}\right| \simeq \frac{4.14\,10^{-15}\,eV\cancel{s}\;3\,10^{8}\,\frac{\cancel{m}}{\cancel{s}}}{\cancel{e}\;10^{-10}\,\cancel{m}} \approx 1.2\,10^{4}\,V$$

A6 Thanks to energy conservation, we have

$$E_3 - E_2 = hf \qquad f = -\frac{G}{h}\left(\frac{1}{9}-\frac{1}{4}\right) \simeq \frac{13.6\,eV}{4.14\,10^{-15}\,eV\,s}\,\frac{5}{36} \simeq 4.6\,10^{14}\,s^{-1}$$

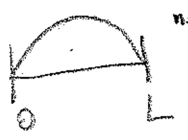
A7 $\frac{h}{\lambda} = p$ and in the non-relativistic approximation we have

$$KE = \frac{p^2}{2m}. \text{ Then } KE = \frac{h^2}{\lambda^2 2m_e} = \frac{(4.14)^2\,10^{-30}\,eV^2\cancel{s^2}\;9\,10^{16}\,\cancel{m^2}}{10^{-20}\,\cancel{m^2}\,2 \cdot 5\,10^{5}\,\cancel{eV}\;\cancel{s^2}} \approx 150\,eV$$

which is much smaller than the rest-mass energy which justifies the use of non-relativistic formulae.

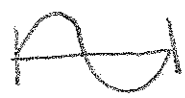
A8 The waves must be zero at the boundarie of the box. Thus we have

$n=1$



or

$n=2$

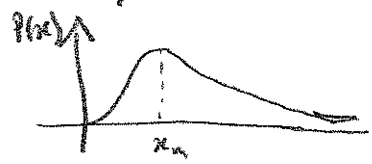


etc. In equations this means

$$n\lambda = 2L. \text{ Then } p = \frac{h}{\lambda} \text{ thus } E = \frac{p^2}{2m} = \frac{h^2 n^2}{8mL^2}$$

A.9 The probability density is $P(x) = |\psi(x)|^2$. Thus

$P(x) \sim x^2 e^{-2x}$. We have



$$\left.\frac{dP}{dx}\right|_{x_m} = 0 \qquad \left(2x - 2x^2\right)e^{-2x} = 0 \Rightarrow x_m = 1$$

A.10 The Heisenberg uncertainty principle states that

$$\Delta x \, \Delta p_x \geq \frac{\hbar}{2}$$ where Δx (or Δp) stays

for the uncertainty on the position (or momentum) _ As the average momentum is clearly zero, then the average KE is

$$\langle KE \rangle = \frac{\Delta p^2}{2m} \geq \frac{\hbar^2}{4} \frac{1}{\Delta x^2} \frac{1}{2m} = \frac{\hbar^2}{8m\,\Delta x^2} \simeq$$

$$\simeq \frac{1}{4\pi^2} \frac{(4.14)^2 \, 10^{-30} \, eV^{\cancel{2}} \cancel{s^2}}{8 \; 5 \; 10^5 \, \cancel{eV} \, \frac{1}{4} 10^{-20} \, \cancel{m^2}} \; 9 \, 10^{16} \frac{\cancel{m^2}}{\cancel{s^2}} \simeq 4 \; eV$$

B1 If $I(f,T)$ denotes the frequency spectrum for the power emitted by unit surface by a black body at temperature T, then

(a)

$$I(T) = \int_0^\infty I(f,T)\, df = \sigma T^4$$

(b) By using Planck's law, we have

$$I(T) = \frac{2\pi h}{c^2} \int_0^\infty \frac{f^3\, df}{e^{\frac{hf}{kT}} - 1} = \frac{2\pi h}{c^2} \left(\frac{kT}{h} \right)^4 \int_0^\infty \frac{x^3\, dx}{e^x - 1}$$

Thus obtaining $I(T) = \sigma T^4$

(c) The energy of each photon is $E = hf$ - Thus the total number of photon is

$$N_{ph} = \frac{2\pi h}{c^2} \int_0^\infty \frac{f^3}{e^{\frac{hf}{kT}} - 1} \frac{df}{hf} = \frac{2\pi h}{c^2} \frac{1}{h} \int_0^\infty \frac{f^2\, df}{e^{\frac{hf}{kT}} - 1}$$

Then the total number of photons emitted N^{ph} is $\sim T^3$.

So the ratio $\dfrac{N^{ph}(T)}{N^{ph}(2T)} = \dfrac{T^3}{(2T)^3} = \dfrac{1}{8} \left(= \dfrac{N_1^{ph}}{N_2^{ph}} \right)$

(d) The intensity per unit wavelength and the one per unit frequency are related as follows $I(\lambda,T)\, d\lambda = |I(f,T)\, df|$. So

$$I(\lambda,T) = \frac{2\pi h}{c^2} \cdot \frac{c^3}{\lambda^3} \frac{1}{e^{\frac{hc}{kT\lambda}}-1} \frac{c}{\lambda^2} d\lambda = \frac{2\pi h c^2}{\lambda^5} \frac{d\lambda}{e^{\frac{hc}{kT\lambda}}-1}$$

f_{max} satisfies $\dfrac{dI(f,T)}{df} = 0 \qquad 3 f_{max}^2 - \dfrac{h}{kT} \dfrac{f_{max}^3}{e^{\frac{hf_{max}}{kT}}-1} = 0$

and λ_{max} " $\dfrac{dI(\lambda,T)}{d\lambda} = 0 \qquad \dfrac{-5}{\lambda_{max}^6} + \dfrac{hc}{kT} \dfrac{1}{\lambda_{max}^7} \dfrac{1}{e^{\frac{hc}{kT\lambda_{max}}}-1} = 0$

Thus $\dfrac{f_{max}}{e^{\frac{hf_{max}}{kT}}-1} = 3$ while $\dfrac{c}{\lambda_{max}} \dfrac{1}{e^{\frac{hc}{kT\lambda_{max}}}-1} = 5$ so $f_{max} \neq \dfrac{c}{\lambda_{max}}$

$\boxed{B2}$ The free (time independent) Schrödinger equation in 1D is

(a) $-\dfrac{\hbar^2}{2m} \dfrac{d^2\varphi(x)}{dx^2} = E\,\varphi(x)$ where in this case we need to impose the boundary conditions $\varphi(x) = \varphi(x+2\pi R)$ $\forall x$.

(b) The most general solution is $\varphi(x) = e^{ipx}$ with $p = \dfrac{\sqrt{2mE}}{\hbar}$

The boundary conditions read $e^{ipx} = e^{ipx + 2\pi i pR}$, which implies $pR = n$, $n \in \mathbb{Z}$.

Then $E_n = \dfrac{\hbar^2 n^2}{2mR^2}$ with $n = 0, \pm 1, \pm 2, \dots$

(c) By indicating with $\psi_n(x)$ the wave function of the n^{th} energy
(d) level, then a 2-particle state is $\psi_n(x_1)\,\psi_m(x_2)$. According to Pauli exclusion principle the total wave function must be anti-symmetric under the exchange $x_1 \leftrightarrow x_2$. Thus the lowest energy state(s) are $\psi_{\pm 1}(x_1)\,\psi_0(x_2) - \psi_0(x_1)\,\psi_{\pm}(x_2)$ with energy $\dfrac{\hbar^2}{2mR^2} = E_{g.f.}$. If they are boson, we have $\psi_0(x_1)\,\psi_0(x_2)$ with zero energy, as the wave function must be symmetric under the exchange $x_1 \leftrightarrow x_2$. Finally the most general state with energy $E_{g.f.}$ in the fermionic case is

$$a\left[\psi_1(x_1)\,\psi_0(x_2) - \psi_0(x_1)\,\psi_1(x_2)\right] + b\left[\psi_{-1}(x_1)\,\psi_0(x_2) - \psi_0(x_1)\,\psi_{-1}(x_2)\right]$$

with $|a|^2 + |b|^2 = 1$.

$\boxed{B3}$ We want to fix A so as to have $\int_0^L |\psi(x)|^2 dx = 1$

(a) $$1 = |A|^2 \int_0^L \left[\cos\left(\frac{2\pi x}{L}\right) - 1\right]^2 dx = |A|^2 \left\{ \int_0^L \cos^2 \frac{2\pi x}{L}\, dx - 2\int_0^L \cos \frac{2\pi x}{L}\, dx + L \right\}$$

$$= |A|^2 \left\{ \frac{1}{2}\int_0^L \left(\cos \frac{4\pi x}{L} + 1\right) dx + L \right\} \quad \text{as} \quad \int_0^L \cos \frac{2\pi n x}{L}\, dx = 0$$

with $n \in \mathbb{Z}$. So $1 = |A|^2 \dfrac{3L}{2}$ and $A = \sqrt{\dfrac{2}{3L}}$

is a possible normalization.

(b) I need to check if $\psi(x)$ satisfies $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$ for some constant E.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = -\frac{\hbar^2}{2m}\frac{d}{dx}\left[-\frac{2\pi}{L}\sin\frac{2\pi x}{L}\right] = +\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}\cos\frac{2\pi x}{L}$$

which is not proportional to the original ψ. So it's not a stationary wave.

(c) The probability of finding the particle between 0 and $\frac{L}{4}$ is

$$\int_0^{L/4}|\psi(x)|^2 dx = \frac{2}{3L}\int_0^{L/4}\left(\cos\frac{2\pi x}{L}-1\right)^2 dx$$ and by using point (a)

$$= \frac{2}{3L}\left\{\int_0^{L/4}\frac{1}{2}\cos\frac{4\pi x}{L}dx + \frac{1}{2}\int_0^{L/4}dx - 2\int_0^{L/4}\cos\frac{2\pi x}{L}dx + \int_0^{L/4}dx\right\}$$

$$= \frac{2}{3L}\left\{0 + \frac{L}{8} - 2\frac{L}{2\pi}\sin\frac{2\pi x}{L}\Big|_0^{L/4} + \frac{L}{4}\right\}$$

$$= \frac{2}{3L}\left\{\frac{3L}{8} - \frac{L}{\pi}\right\} = \frac{1}{4} - \frac{2}{3\pi}$$

(d) The average kinetic energy is $\langle KE\rangle = \int_0^L \bar{\psi}(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi(x)\,dx$

$$\langle KE\rangle = \frac{2}{3L}\int_0^L\left(\cos\frac{2\pi x}{L}-1\right)\left(\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2}\right)\cos\frac{2\pi x}{L} =$$

$$= \frac{2}{3L}\left\{\int_0^L\cos^2\left(\frac{2\pi x}{L}\right)dx - \int_0^L\cos\left(\frac{2\pi x}{L}\right)\right\}\frac{\hbar^2}{2m}\frac{4\pi^2}{L^2} = \frac{1}{3}\frac{\hbar^2 4\pi^2}{2mL^2}$$

$\boxed{B4}$ The ionization energy is $\dfrac{hc}{\lambda_{n=\infty}}$. From Rydberg

(a) formula and $\lambda_{n=2} = 1.2\ 10^{-7}\,m$ we have

$$\frac{1}{\lambda_{n=2}} = R\left(1-\frac{1}{4}\right) = \frac{3R}{4}, \quad \text{so}$$

$$\frac{hc}{\lambda_{n=\infty}} = hcR = \frac{4}{3}\frac{hc}{\lambda_{n=2}} = \frac{4}{3}\frac{4.14\,eVs\ 3\ 10^{8}\,\frac{m}{s}}{1.2\ 10^{-7}\,m} \simeq 13.8\,eV$$

(b) The classical equation of motion of the electron is $F = ma$

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m\frac{v^2}{r} \quad \text{(assuming uniform circular motion).}$$

Quantization of the angular momentum reads $mvr = n\hbar$ (in Bohr's model)

$$v_n = \frac{1}{4\pi\epsilon_0}\frac{e^2}{n\hbar} \quad \text{and} \quad r_n = \frac{n\hbar}{mv_n} = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2}$$

Thus the total energy is $\frac{1}{2}mv_n^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = E_n$

$$E_n = \frac{1}{2}m\frac{1}{(4\pi\epsilon_0)^2}\frac{e^4}{n^2\hbar^2} - \frac{1}{4\pi\epsilon_0}\frac{me^4}{4\pi\epsilon_0 n^2\hbar^2} = -\frac{me^4}{32\pi^2\epsilon_0^2 n^2\hbar^2}$$

(c) By energy conservation we have $\dfrac{hc}{\lambda_n} = E_n - E_1$. So

$$\frac{1}{\lambda_2} = \frac{me^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{hc}\left(1-\frac{1}{n^2}\right) \quad \text{which yields}$$

$$R = \frac{me^4}{8\epsilon_0^2 h^3 c} \approx \frac{5\,10^5 \frac{eV}{c^2}\, e^4}{8\left(\frac{e^2}{2hc}\,137\right)^2 h^3 c} = \frac{5 \cdot 10^5\, eV}{2hc\, 137^2}$$

$$\approx \frac{5 \cdot 10^5\, \cancel{eV}}{2\; 4.14\; 10^{-15}\, \cancel{eV}\cancel{s}\; 3\,10^8 \frac{m}{\cancel{s}}} \frac{1}{(137)^2} \approx 1.08\; 10^7 \frac{1}{m}$$

(d) In the case of a Helium ion the equation of motion reads

$$m\frac{v^2}{r} = \frac{2e^2}{r^2}$$ with the same quantization law of L

Then $v_n^{He} = 2v_n$ and $r_n^{He} = \frac{r_n}{2}$ Thus $E_n^{He} = 4E_n$ and

$R^{He} = 4R$

QUANTUM PHYSICS 2010

A1

The relativistic kinetic energy is

$$T = (\gamma - 1)\, m c^2 \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

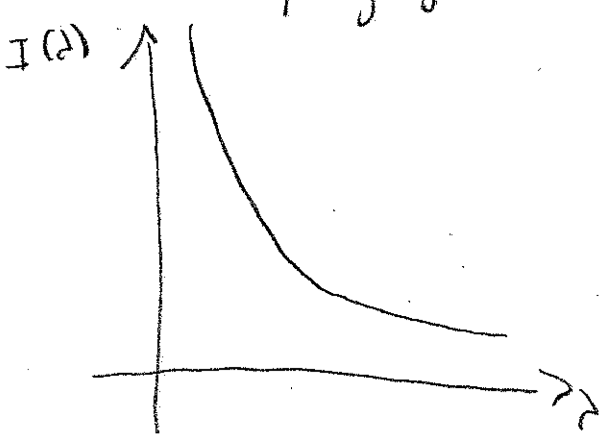
By taking the ratio of T and the rest mass energy we get

$$\frac{T}{mc^2} = (\gamma - 1) = \frac{100\ \text{MeV}}{100\ \text{MeV}} = 1 \quad \Rightarrow \quad \gamma = 2$$

Thus the non-relativistic approximation is NOT appropriate

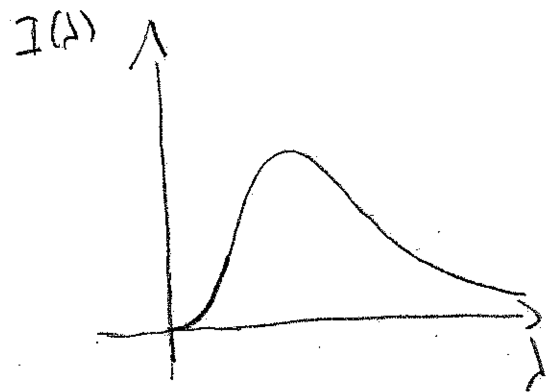
A2

Rayleigh-Jeans



$$I(\lambda) \sim \frac{KT}{\lambda^4}$$

Planck's law



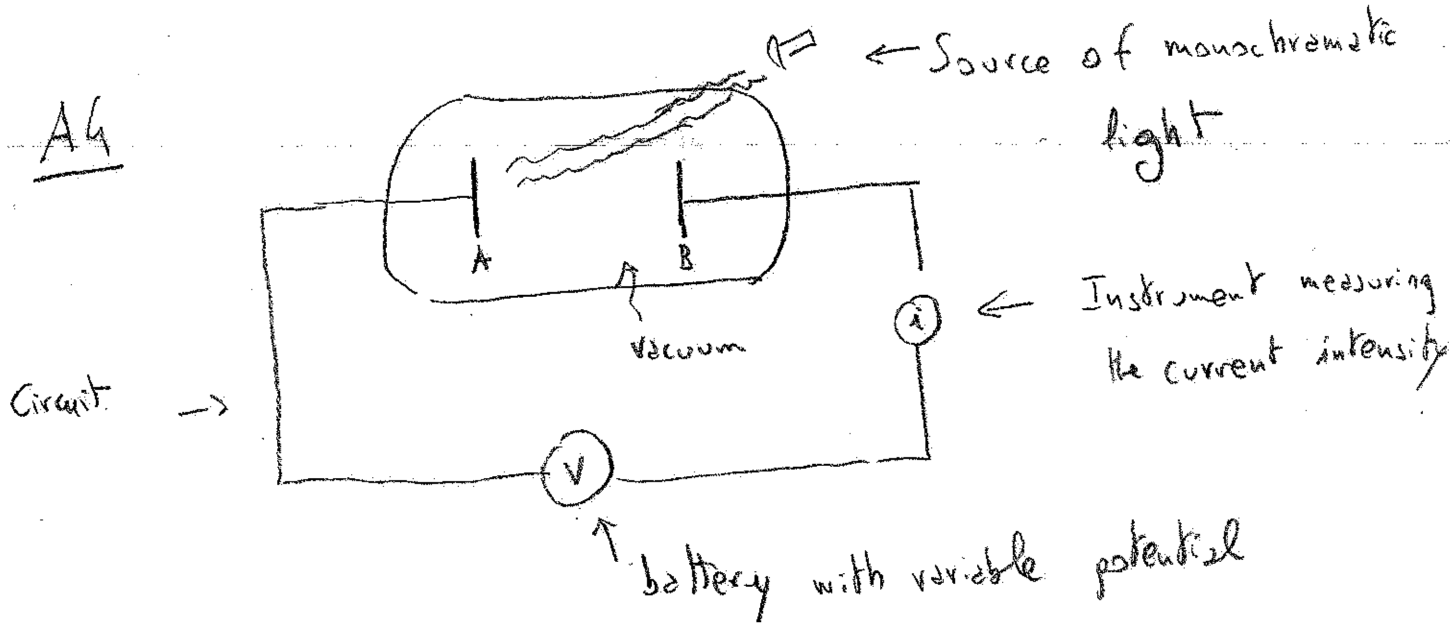
$$I(\lambda) \sim \frac{1}{\lambda^5} \frac{\lambda}{e^{\frac{hc}{\lambda KT}} - 1}$$

where $I(\lambda)$ is the intensity per unit wavelength of the emitted radiation

A3 The relation between the spacing of the bright fringes Δ, the wavelength λ, the slit separation d and the screen distance R is

$$\lambda = \frac{\Delta d}{R} \Rightarrow \lambda = \frac{1.5\,10^{-3}\,m \;\; 4.10^{-3}\,m}{10.\,m} = 6\,10^{-7}\,m$$

A4



The stopping potential V_s is the potential the battery needs to provide to stop the current flowing in the circuit. This means that electrons are stopped before reaching plate B because the potential energy is bigger that their initial kinetic energy. This kinetic energy depends on the energy of each incident photons ($E_p = hf$) and thus is related to the frequency, and not to the intensity (which measures the number of photons in the beam).

A5 The electrons are emitted if

$$\frac{hc}{\lambda} > 1.9\,eV \qquad \frac{4.14\ 10^{-21}\ MeVs\ 3\cdot 10^{8}\frac{m}{s}}{2.5\ 10^{-7}\ m} \approx 5\,eV$$

So they are emitted. Their kinetic energy is

$$KE = \frac{hc}{\lambda} - \phi = 5\,eV - 1.9\,eV \approx 3.1\,eV$$

A6 The minimal wavelength is

$$\frac{hc}{\lambda_{min}} = 10^{4}\,eV \qquad \lambda_{min} = \frac{4.14\ 10^{-15}\ 3\cdot 10^{8}}{10^{4}}\ m = 12.4\ 10^{-11}\ m$$

A7 According to de Broglie we have $p = \frac{h}{\lambda}$.

In our case $p = \frac{4.14\,10^{-15}\,eVs}{10^{-10}\,m} = 4.14\,10^{-5}\,eV\frac{s}{m}$ -

If I use non-relativistic formulae, I have for the kinetic energy

$$T = \frac{p^2}{2m} = \frac{(4.14\ 10^{-5})^2\,eV^2\,\frac{s^2}{m^2}}{2\ 5\,10^{5}\ \frac{eV}{c^2}} \approx 1.6\ 10^{-15}\ 9\ 10^{16}\ eV = 150\,eV$$

which is much smaller than the rest mass energy $5\,10^{5}\,eV$.

A8 By conservation of energy

$$E_2 - E_1 = \frac{hc}{\lambda} = \frac{4.14\ 10^{-15}\ eV_s \cdot 3 \cdot 10^{8} \frac{m}{s}}{6.5\ 10^{-7}\ m} \approx 1.9\ eV$$

A9 We should normalise the wavefunction by requiring $\int_0^\lambda |\psi(x)|^2 dx = 1$. In our case

$$\int_0^\lambda |\psi(x)|^2 dx = |N|^2 \int_0^\lambda dx = |N|^2 \quad \text{so}$$

any choice $N = e^{i\vartheta}$ with $\vartheta \in \mathbb{R}$ does the job.

The probability density $P(x) = |\psi(x)|^2$ is just constant

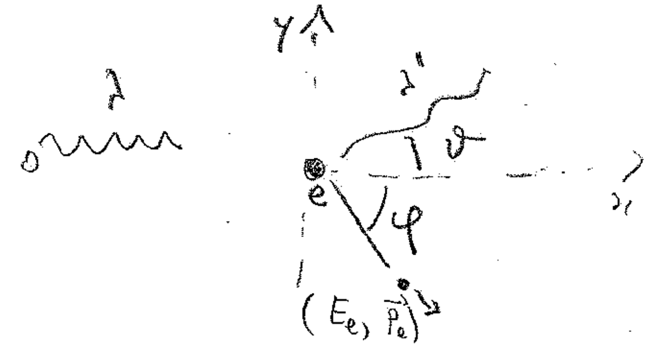
A.10 The product of the position (Δx) and momentum uncertainties (Δp) should satisfy $(\Delta x)(\Delta p) \geq \frac{\hbar}{2}$.

Δp is defined as $\Delta p^2 = \langle (p - \langle p \rangle)^2 \rangle$ and for a confined particle $\langle p \rangle$ is clearly zero. Thus

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{\Delta p^2}{2m} \geq \frac{\hbar^2}{8 m_p \Delta x^2} \approx 5\ MeV$$

B1

(i)



λ wavelength before the scattering

λ' wavelength after the scattering

$(E_e, \vec{p}_e)$ energy and momentum of the electron after the scattering

(1) $$\frac{hc}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + E_e$$

(2a) $$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\vartheta + (p_e)_x$$

(2b) $$0 = \frac{h}{\lambda'} \sin\vartheta - (p_e)_y$$

(1) is Energy conservation while (2a) and (2b) represents momentum conservation (along the x and y axis).

(ii) We can use the relation $E^2 - p^2c^2 = m^2c^4$ to get rid of $\vec{p}_e$

$$E_e = \left(\frac{hc}{\lambda} + m_e c^2 - \frac{hc}{\lambda'}\right)$$

$$\vec{p}_e = \frac{h}{\lambda}\vec{e}_x - \frac{h}{\lambda'}\vec{e}_\vartheta$$

where $\vec{e}_x$ is a versor along the x-axis and $\vec{e}_\vartheta$ a versor at an angle ϑ from $\vec{e}_x$.

$$E_e^2 - p_e^2c^2 = m_e^2c^4 = \left(\frac{hc}{\lambda} + m_e c^2 - \frac{hc}{\lambda'}\right)^2 - \frac{h^2}{\lambda^2}c^2 - \frac{h^2}{\lambda'^2}c^2 + \frac{2h^2\cos\vartheta\, c^2}{\lambda\lambda'}$$

$$0 = 2m_e c^2\left(\frac{hc}{\lambda} - \frac{\lambda c}{\lambda'}\right) - 2\frac{h^2c^2}{\lambda\lambda'}(1-\cos\vartheta)$$

5

$$\frac{1}{\lambda} - \frac{1}{\lambda'} = \frac{h}{m_e c} \frac{1-\cos\vartheta}{\lambda\lambda'} \Rightarrow \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\vartheta)$$

(iii) From momentum conservation, I have

$$(2b) \Rightarrow \frac{h}{\lambda'} \sin\vartheta = (p_e)_y = |p_e| \sin\varphi \overset{\text{in our case}}{\Rightarrow} \frac{h}{\lambda'} = |p_e|$$

$$(2a) \Rightarrow \frac{h}{\lambda} = \frac{h}{\lambda'} \cos\vartheta + |p_e| \cos\varphi \Rightarrow \frac{h}{\lambda} = \frac{h}{\lambda'} \sqrt{3} \Rightarrow \lambda' = \sqrt{3}\lambda$$

Finally from Compton's formula we have

$$\lambda' - \lambda = \lambda(\sqrt{3}-1) = \frac{h}{m_e c}\left(1 - \frac{\sqrt{3}}{2}\right) \Rightarrow \lambda = \frac{h}{m_e c} \frac{2-\sqrt{3}}{2(\sqrt{3}-1)}$$

B2

(i) $$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} = E\,\psi(x)$$ with boundary conditions

$$\psi(x=0) = \psi(x=L) = 0$$

(ii) $\psi(x) = A \sin \frac{n\pi x}{L}$ satisfies the above differential equation if $E = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$

and also the boundary conditions if

$$\sin \frac{n\pi L}{L} = 0 \Rightarrow n \text{ is integer.}$$

We focus on the values $n = 1, 2, 3, \ldots$ since

$n=0$ just yields the trivial solution $(\psi(x)=0)$ and the negative values are equivalent to the positive ones (by an immaterial change of sign of $\psi(x)$) –

Thus the allowed energy levels are $E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$ with $n \in \mathbb{N}$

(iii) The ground state is

$$\psi(x) = A \sin \frac{\pi x}{L}$$ and is normalised by requiring

$$\int_0^L |\psi(x)|^2 dx = 1 \Rightarrow 1 = |A|^2 \int_0^L \sin^2 \frac{\pi x}{L}\, dx = |A|^2 \frac{L}{2}$$

Then the average position is

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2 \frac{\pi x}{L}\, dx = \frac{2}{L} \int_0^L \left(x - \frac{L}{2}\right) \sin^2 \frac{\pi x}{L}\, dx + \int_0^L \sin^2 \frac{\pi x}{L}$$

the first integral vanishes since it is a product of an antisymmetric function (under $x \to L-x$) and a symmetric one.

Thus $$\langle x \rangle = \int_0^L \sin^2 \frac{\pi x}{L}\, dx = \frac{L}{2}$$

(iv) Bosons $\rightarrow$ $$E_{TOT} = 2\, \frac{\hbar^2 \pi^2}{2mL^2}$$

Fermions $\rightarrow$ $$E_{TOT} = \frac{\hbar^2 \pi^2}{2mL^2}(1+2^2) = \frac{5 \hbar \pi^2}{2mL^2}$$

$\boxed{B3}$

(i) $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}\kappa x^2\psi(x) = E\psi(x)$$

(ii) $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} e^{-\frac{\sqrt{\kappa m}}{2\hbar}x^2} = -\frac{\hbar^2}{2m}\frac{d}{dx}\left(-2\frac{\sqrt{\kappa m}}{2\hbar} x\, e^{-\frac{\sqrt{\kappa m}}{2\hbar}x^2}\right)$$

$$= -\frac{\hbar^2}{2m}\left(-\frac{\sqrt{\kappa m}}{\hbar} + 4\frac{\kappa m}{4\hbar^2}x^2\right)e^{-\frac{\sqrt{\kappa m}}{2\hbar}x^2}$$

$$= \left[+\frac{\hbar}{2}\sqrt{\frac{\kappa}{m}} - \frac{1}{2}\kappa x^2\right] e^{-\frac{\sqrt{\kappa m}}{2\hbar}x^2}$$

By plugging this result in the Schrödinger equation at the point (i) we see that the terms proportional to x^2 cancel and the equation is satisfied if $E = \frac{\hbar}{2}\sqrt{\frac{\kappa}{m}}$

(iii) $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \left(\frac{1}{2}\kappa x^2 - Fx\right)\psi(x) = E\psi(x)$$

where I used the relation between force and potential energy $F = -\frac{dU}{dx}$ when F is constant $\Rightarrow U = -Fx$

(iv) The equation in iii can be rewritten as follows

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}\kappa\left(x - \frac{F}{2\kappa}\right)^2\psi(x) = \left(E + \frac{F^2}{4\kappa^2}\right)\psi(x)$$

Then by introducing $y = x - \frac{F}{2k}$ we have that in terms of y the Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(y)}{dy^2} + \frac{1}{2}ky^2\psi(y) = \left(E + \frac{F}{4k^2}\right)\psi(y)$$

implying that $\psi(y)$ takes the same form as in (ii) and thus $\psi(x)$ reads as

$$\psi(x) = A\exp\left[-\frac{\sqrt{km}}{2\hbar}\left(x - \frac{F}{2k}\right)^2\right]$$

B4 The Coulomb force is $|F| = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$

In Bohr's model the electron follows a circular trajectory so its acceleration is $a = \frac{v^2}{r}$. Thus we have

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} = m_e\frac{v^2}{r} \Rightarrow v^2 = \frac{1}{4\pi\epsilon_0 m_e}\frac{e^2}{r}$$

(ii) The classical energy is $E = K + U$

$$E = \frac{1}{2}m_e v^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} = \frac{1}{2}\cancel{m_e}\frac{1}{4\pi\epsilon_0\cancel{m_e}}\frac{e^2}{r} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$= -\frac{1}{8\pi\epsilon_0}\frac{e^2}{r},$$

while the angular momentum L is

$$L = m_e v r = m_e\sqrt{\frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e r}}\,r = \frac{e\sqrt{m_e r}}{\sqrt{4\pi\epsilon_0}}$$

(iii) Bohr's quantisation states $|L| = m_e v r = n\hbar$

Thus by using the relation between v and r we have

$$v^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e r} \Rightarrow v = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e v r} = \frac{e^2}{4\pi\epsilon_0 \hbar n}$$

$$\Rightarrow r = \frac{e^2}{4\pi\epsilon_0 m_e}\frac{(4\pi\epsilon_0)^2\hbar^2 n^2}{e^4} = \frac{4\pi\epsilon_0 \hbar^2 n^2}{m_e e^2}$$

Thus $$E = -\frac{1}{8\pi\epsilon_0}\frac{e^2 m_e e^2}{4\pi\epsilon_0 \hbar^2 n^2} = -\frac{1\, m_e e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} \approx 13.6\,\text{eV}$$

(iv) Conceptually the derivation works as before with two changes

→ we need to use the mass of the muon instead of that of the electron

→ the nucleus will have charge $k e l$ since it's an α-particle

Thus $$V = \frac{2e^2}{4\pi\epsilon_0\hbar n} \Rightarrow E = -\frac{m_\mu 4 e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} \approx 13.6 \cdot 4\ 200\ \text{eV} \approx 10^4\,\text{eV}$$