QUANTUM MECHANICS A (SPA-5319)

Tunneling Through an Arbitrary Barrier

Consider a rectangular barrier of height V_0 being approached from the left by a particle with energy $E < V_0$, so that classically it cannot pass through:

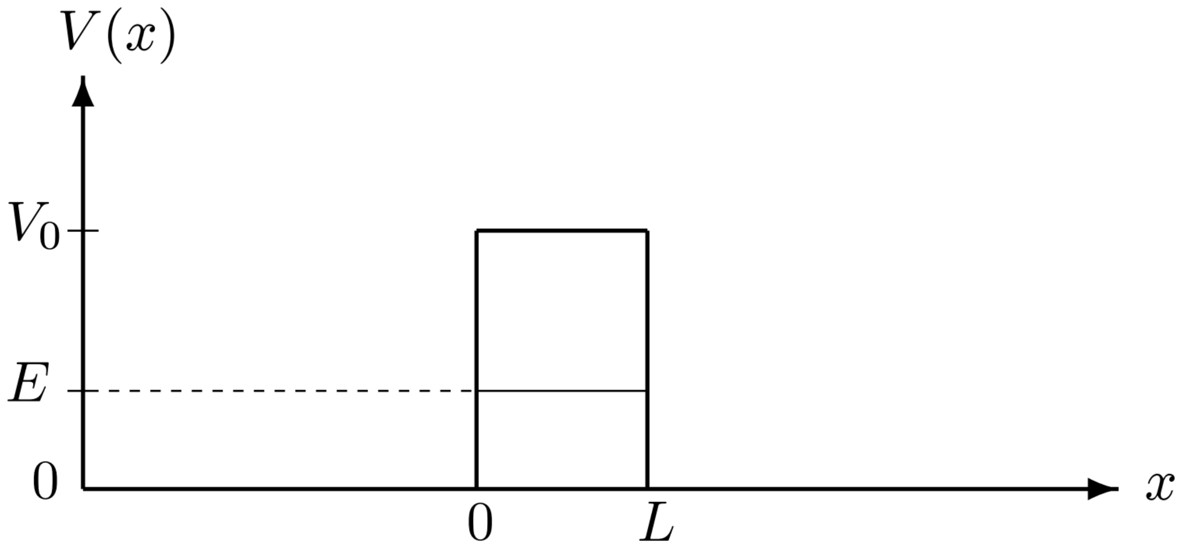

Figure 1: *A free particle is incident on a rectangular barrier with energy less than the height of the barrier, $E < V_0$.*

In quantum mechanics we have to find the wave function by solving the TISE outside and inside the barrier. Outside, $V = 0$ and we take the wave function to be a plane de Broglie wave. Inside the barrier we have $V = V_0 > E$, so the TISE is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\psi = E\psi,$$

which can be rearranged in the usual way to give,

$$\begin{aligned}\frac{d^2\psi}{dx^2} &= +\left[\frac{\hbar^2}{2m}(V_0 - E))\right]\psi \\ &= +\kappa^2\psi \qquad\qquad \text{where,} \quad \kappa = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}\end{aligned}$$

As $(V_0 - E)$ is positive, κ is real, and the equation has exponential rather than oscillatory solutions:

$$\psi = Ae^{-\kappa x} + Be^{+\kappa x}$$

It can be shown that for high and wide barriers ($\kappa L \gg 1$) the second term can be neglected (see e.g. Bransden & Joachain, p.150) and so the wavefunction has the simple form of a decaying exponential inside the barrier. Even for a wide barrier the exponential has not decayed to zero when we reach the end of the barrier, $x = L$. The Born interpretation then immediately tells us that the probability of finding the particle on the other side of the barrier is not zero: the probability that the particle tunnel through from $x = 0$ to $x = L$ is

$$T_{0\to L} = |\psi(x = L)|^2 \propto e^{-2\kappa L} \qquad \text{where} \quad \kappa = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

We recognise this as the earlier approximation obtained for the classically strongly forbidden case; here our derivation was simpler, but a bit less honest – a quick fix!

Continuity of the wave function, of course, requires that the free transmitted de Broglie wave joins on smoothly to the exponential tail at $x = L$ with much smaller amplitude than the incident one: To extend our theory to barriers of any shape (see Figure 2) we simply break up the potential into narrow rectangular strips, the i-th one having width L_i and height V_i. Thus, if the particle has succeeded in tunneling as far as the i-th strip, the probability that it then tunnel through this strip is approximately,

$$T_i \propto e^{-2\kappa_i L_i} \qquad \text{where,} \quad \kappa_i = \sqrt{\frac{\hbar^2}{2m}(V_i - E)}$$

Thus, the probability for a particle with energy E to traverse the entire barrier from $x = a$ to $x = b$, is the product of the independent probabilities for traversing each successive rectangular forbidden barrier:

$$T_{a\to b} = T_1 T_2 \dots T_i \dots T_N$$

$$T \propto e^{-2\Sigma_{i=1}^{N}\kappa_i L_i}$$

where we have used the fact that the exponents add in a product. Finally in the limit where the strips become infinitesimal, $L_i \to dx$, $N \to \infty$, the label i becomes the continuous label x, i.e. $V_i \to V(x)$, and the sum becomes an integral, $\Sigma_i \to \int dx$. This gives

$$T(E) = Ae^{-G(E)} \text{ where the Gamow factor is } G(E) = 2\left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \int_a^b \sqrt{(V(x) - E)}\, dx$$

This derivation of Gamow's famous formula has been rather cavalier, but is essentially correct: the approximation is known as the WKB approximation (Wentzel, Kramers, Brillioun, who were the originators of the basic method), and applies to potentials that vary slowly enough; it is an example of a semi-classical approximation although tunneling itself is very far from a classical phenomenon. The proportionality constant A is, in fact, a slowly varying function of E, $A = A(E)$.

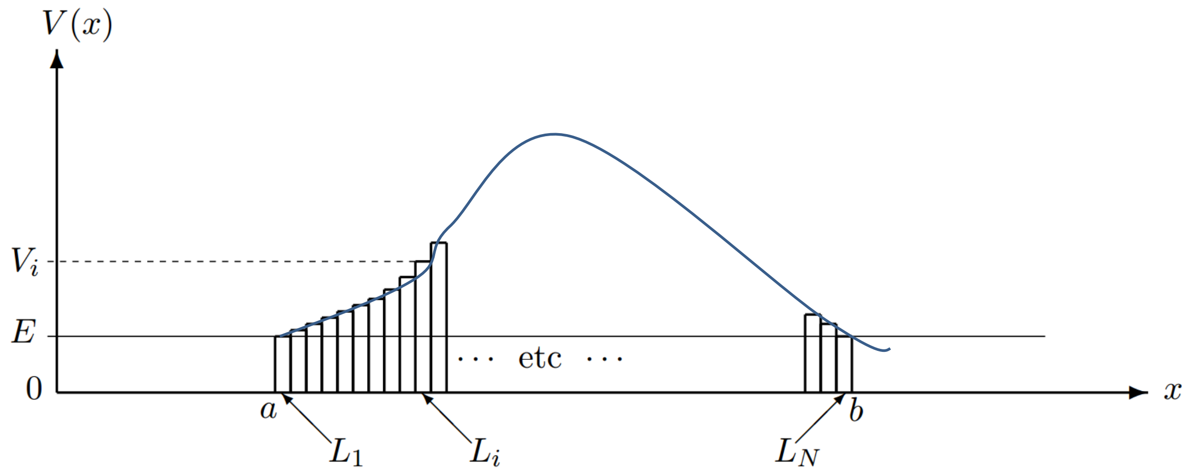

Figure 2: potential barrier broken up into N narrow rectangular barriers of width L_i and height V_i. The region $a \rightarrow b$ is classically forbidden to the particle with energy E, where $V(a) = V(b) = E$. The points a and b are the so-called classical turning points.