QUANTUM MECHANICS A (SPA 5319)

Parity

Most of the examples of quantum wells you will meet in this course have energy eigenfunctions with a definite symmetry under the transformation $x \to -x$: they are either symmetric or antisymmetric:

$$\text{SYMMETRIC} \qquad (\text{Parity } +1): \qquad \Psi_E(-x) = +\Psi_E(x) \tag{1}$$

$$\text{ANTISYMMETRIC} \qquad (\text{Parity } -1): \qquad \Psi_E(-x) = -\Psi_E(x) \tag{2}$$

Thus for the infinite square well, with the origin at the centre of the well, and for $-L/2 \leq x \leq L/2$:

$$\text{The symmetric states are} \quad \Psi_E(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 3, 5, ... \tag{3}$$

$$\text{The antisymmetric ones are } \Psi_E(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad n = 2, 4, 6, ... \tag{4}$$

Similarly for the finite square well and, as we shall see later, for the harmonic oscillator potential $V(x) = kx^2/2$. Inspection of these potentials reveals that they are all symmetric about the appropriately chosen origin of coordinates,

$$V(-x) = V(x) \tag{5}$$

We now proceed to show that this is the key ingredient.

THEOREM: For all potentials that are symmetric under reflections about some point, the energy eigenfuntions have definite parity: they are either symmetric or antisymmetric under reflection of the axes about the symmetry point of the potential[1],

$$\Psi(-x) = \pm\psi(x) \tag{6}$$

This is the simplest example of one of the most important concepts in physics: symmetry under some transformation - here a mirror reflection about a suitably chosen origin - leads to a conservation law - here the conservation of the symmetry (or parity) of energy eigenstates.

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[1]The symmetry of the wave functions is only obvious when we choose the origin to be the point about which the potential is symmetric. If we were to choose the origin of the infinite square well problem to be at the left wall of the potential the eigenfunctions would be $\Psi_E(x) = \sin n\pi x/L$, $n = 1, 2, 3, 4, 5, 6, ...$ for $0 \leq x \leq L$ and $\Psi_E(x) = 0$ for $x < 0$ and $> L$. This is clearly neither symmetric nor antisymmetric about $x = 0$ (draw the first two and see), but is about $x = L/2$, the mid-point (or symmetry point) of the potential.

PROOF: As $V(-x) = V(x)$, the physics must be the same for x and $-x$. In quantum mechanics all physical observables depend not on $\Psi(x)$ alone, but on $|\Psi(x)|^2$. Therefore the symmetry of $V(x)$ implies that $|\Psi(x)|^2$ shares the same symmetry:

$$|\Psi(-x)|^2 = |\Psi(x)|^2 \tag{7}$$

Therefore $\Psi(x)$ and $\Psi(-x)$ can only differ by a constant (complex) number, λ, with modulus one[2]:

$$\Psi(-x) = \lambda\Psi(x) \tag{8}$$

with

$$|\lambda|^2 = \lambda^*\lambda = 1 \tag{9}$$

Now change variable to $y = -x$. This gives

$$\Psi(y) = \lambda\Psi(-y) \tag{10}$$

Next, change the name of this variable y to x:

$$\Psi(x) = \lambda\Psi(-x) \tag{11}$$

Finally, substitute for $\Psi(-x)$ in this equation using equation (8)[3]:

$$\Psi(x) = \lambda^2\Psi(x) \tag{12}$$

Hence,

$$\lambda^2 = 1 \tag{13}$$

leading to our final result:

$$\lambda = \pm 1 \tag{14}$$

So, the only possibilities are

$$\Psi(-x) = \pm\Psi(x) \tag{15}$$

i.e. for a symmetric potential the energy eigenstates are either symmetric (even parity, or parity +1) or antisymmetric (odd parity, or parity -1).

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[2] In fact, since λ is the most general complex number with unit modulus, it must have the form of a phase factor, $\lambda = e^{i\alpha}$. This is because $|e^{i\alpha}|^2 = e^{-i\alpha}e^{i\alpha} = 1$. Also notice that the condition $|\lambda|^2 = \lambda^*\lambda = 1$ is not the same as $\lambda^2 = 1$ because $\lambda = e^{i\alpha}$ is still a complex number - we will now prove that in fact λ is real.

[3] This result simply amounts to saying that if you reflect twice, $x \to -x \to x$, there is no change