QUANTUM MECHANICS A (SPA-5319)

Alpha Decay & Tunneling

This was one of the first applications of quantum mechanics, made independently in 1928 by Gamow and Condon & Gurney. First some experimental facts about α-decay:

A nucleus with positive charge $Ze = +(Z_1 + Z_2)e$ decays to a daughter nucleus of charge $+Z_1 e$ by the emission of a positively charged α-particle, $Z_2 = 2$. Measured half-lives span an enormous range from $\tau_{1/2} \sim 10^{-7}$ sec. to 5×10^{15} yrs. Emerging α-particle energies are typically a few MeV, 5 MeV being a representative value used in our subsequent estimates.

$^{226}\text{Ra}_{88}$	$E_\alpha = 4.78$ & 4.59 MeV	$\tau_{1/2} = 1.6 \times 10^3$ yrs.
$^{238}\text{U}_{92}$	$E_\alpha = 4.18$ MeV	$\tau_{1/2} = 4.5 \times 10^9$ yrs.
$^{239}\text{Pu}_{94}$	$E_\alpha = 5.15$ MeV	$\tau_{1/2} = 2.4 \times 10^4$ yrs.

(The notation for nuclei is that the superscipt is the mass number (A) and the subscript is the nuclear charge (Z) in units of e). In a nucleus the Coulomb repulsion of the positive constituents is dominated by an overwhelmingly powerful attractive nuclear force that acts only over a very short range ≤ 1 fm $= 10^{-15}$ m. The radii of nuclei range from $R \sim 1 \times 10^{-15} \rightarrow 5 \times 10^{-15}$ m. We will model the approximately spherically symmetric potential felt by the α-particle bound in the nucleus by a one-dimensional deep square well of size R. Beyond R the α-particle does not experience the nuclear force; instead it is subject to the Coulomb repulsion of the daughter nucleus:

$$V(x) = -V_0 \qquad \text{for} \quad 0 < x < R \tag{1}$$
$$= \frac{Z_1 Z_2 e^2}{x} \qquad \text{for} \quad x \geq R \tag{2}$$

The diagram in figure 1 shows this situation. In 3-dimensions the potential would actually be rotated about the vertical axis, but this is a 1-dimensional model of that situation, hence the absence of a region to the left of the origin. The energy levels in the nucleus are discrete, and can be determined from laboratory experiments. The observation that typical emergent α-particle energies are ~ 5 MeV tells us the height of the energy level it occupies: 5 MeV above the zero. Classically the α-particle must climb up to the top of the barrier, requiring it to expend energy $(V_{max} - 5$ MeV); then it would 'roll' down the right-hand side of the Coulomb barrier (being repelled by the daughter nucleus), regaining at a distance b all the energy it expended in climbing to V_{max}. The problem with α-decay is that V_{max} is very large, and there is nowhere the α-particle could obtain so much energy within the nucleus. Where then does the energy come from? The quantum mechanical answer is that the particle does not climb over the potential barrier but instead tunnels through it.

(1) First we calculate the height of the barrier, which is the value of the Coulomb potential at the nuclear radius:

$$V_{max} = V(R) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 R} = 1.44\, Z_1 Z_2 \quad \text{MeV} \tag{3}$$

using $R = 1 \times 10^{-15}$ m. For the case of $^{239}\text{Pu}_{94}$, $Z_1 = 92$, $R \sim 5 \times 10^{-15}$ m, so that, with $Z_2 = 2$, $V_{max} = 50$ MeV.
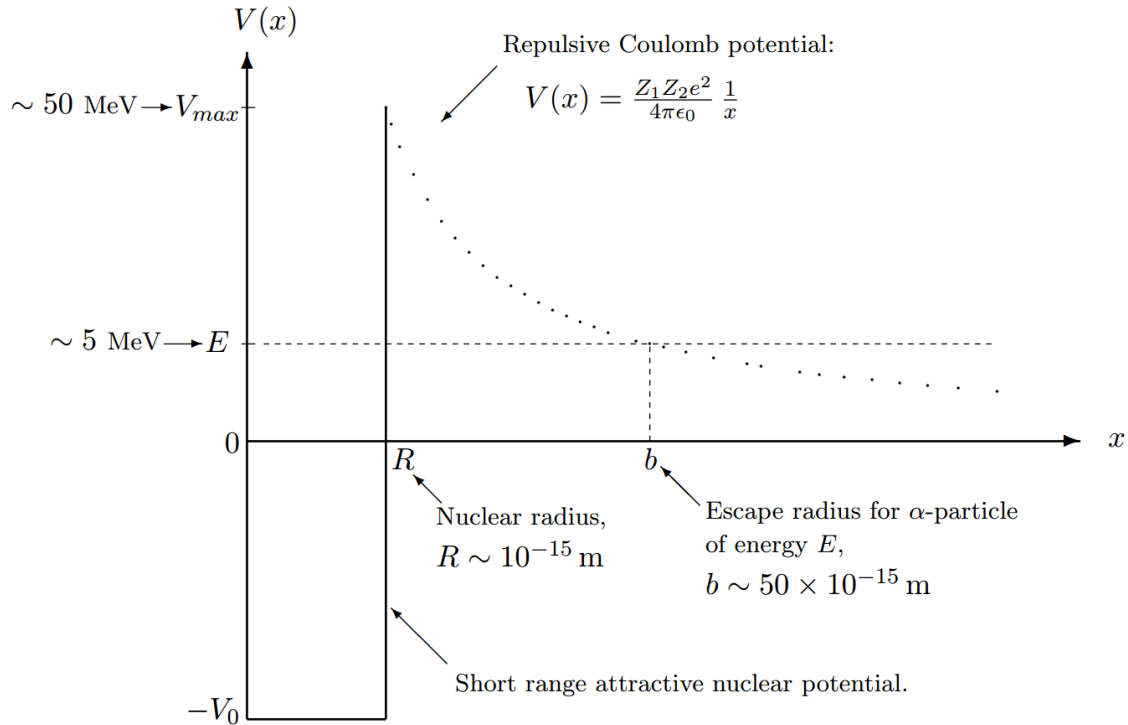

Figure 1: *Potential experienced by α-particle inside & outside a nucleus. (Not to scale!)*

(2) The position, $x = b$, at which the α-particle emerges with the energy E it had in the nucleus is given by

$$E = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0\, b} = \frac{1.44\, Z_1 Z_2}{b(\text{in units of } 10^{-15}\,\text{m})} \quad \text{MeV} \tag{4}$$

leading to

$$b = \frac{1.44\, Z_1 Z_2}{E(\text{in MeV})} \times 10^{-15} \quad \text{m.} \tag{5}$$

With $E \sim 5$ MeV for plutonium we find $b \sim 53 \times 10^{-15}$ m, some ten times further out than the nuclear radius.

Thus we see the difficulties in this way of explaining α-decay: the very large height and width of the barrier seen by the α-particle. This situation is just the one for which the approximation sketched out in the Appendix applies, with tunnelling probability

$$T(E) \simeq A(E)\, e^{-G(E)} \tag{6}$$

with $A(E)$ a slowly varying function of energy compared with the dominant energy-dependent tunnelling exponential determined by the Gamow factor,

$$G(E) = 2 \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \int_R^b \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\frac{1}{x} - E\right)^{\frac{1}{2}} dx \tag{7}$$

where we have used the Coulomb potential for $V(x)$ in the region outside the nucleus. Writing E in terms of b makes the two terms in the integrand of similar form, giving

$$G(E) = 2 \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right)^{\frac{1}{2}} \int_R^b \left(\frac{1}{x} - \frac{1}{b}\right)^{\frac{1}{2}} dx \tag{8}$$

Evaluating the integral: $I = \int_R^b \left(\frac{1}{x} - \frac{1}{b}\right)^{\frac{1}{2}} dx = \int_R^b \left(1 - \frac{x}{b}\right)^{\frac{1}{2}} \frac{dx}{x^{\frac{1}{2}}} = 2b^{\frac{1}{2}} \int_{\left(\frac{R}{b}\right)^{\frac{1}{2}}}^1 (1 - y^2)^{\frac{1}{2}} dy$

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In the last step we changed variable of integration to: $y = \left(\frac{x}{b}\right)^{\frac{1}{2}}$, giving $\frac{dx}{x^{\frac{1}{2}}} = 2b^{\frac{1}{2}}\, dy$.

Now, although this can be evaluated exactly, there is no need to do so because the lower limit is sufficiently small ($R/b \sim 1/10$ or less) that we may approximate it by 0. The integral is then $\pi/4$, the area under the first quadrant of a circle of unit radius. Hence,

$$G(E) \simeq 2 \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right)^{\frac{1}{2}} \frac{\pi\sqrt{b}}{2} \simeq \pi \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right) \frac{1}{\sqrt{E}} \simeq c\, \frac{Z_1 Z_2}{\sqrt{E}} \tag{9}$$

where c is a constant. Since the decay rate is proportional to the probability of tunneling, which is the transmission coefficient, and the half-life is inversely proportional to the decay rate, we obtain[1]

$$\tau_{1/2} \propto \frac{1}{T(E)} = \frac{1}{A(E)} e^{+c\, Z_1 Z_2/\sqrt{E}} \tag{10}$$

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[1]   A very rough estimate of the proportionality constant can be made by imagining the α-particle as bouncing back-and-forth in the nucleus with speed v. Thus at intervals of order $2R/v$ seconds it hits the barrier. As the probability that it tunnels is e^{-G}, the average number of encounters needed before escape is e^{+G}, and therefore the total time required for escape is $\tau \approx (2R/v)e^{+G}$. The speed is that corresponding to the $E \approx 5$ MeV α-particle, $v \approx \sqrt{2E/m} \approx 2 \times 10^7$ m/sec. See Appendix A for further details.

Taking logs, and remembering that $A(E)$ is sufficiently slowly varying with E that we may assume it is approximately constant, we obtain

$$\log\tau_{1/2} \simeq \text{const} + \frac{cZ_1Z_2}{\sqrt{E}} \qquad (11)$$

Thus we have a prediction for the energy-dependence and Z_1-dependence (daughter nucleus charge; remember that $Z_2 = 2$ for α-decay) of the half-life that can be compared with experiment over an enormous range of scales. The accompanying diagram shows how well the experimental points are fitted with a straight line in the $\log\tau_{1/2} - Z_1/\sqrt{E}$ plot, providing strong evidence that the decays do indeed proceed by quantum tunnelling.
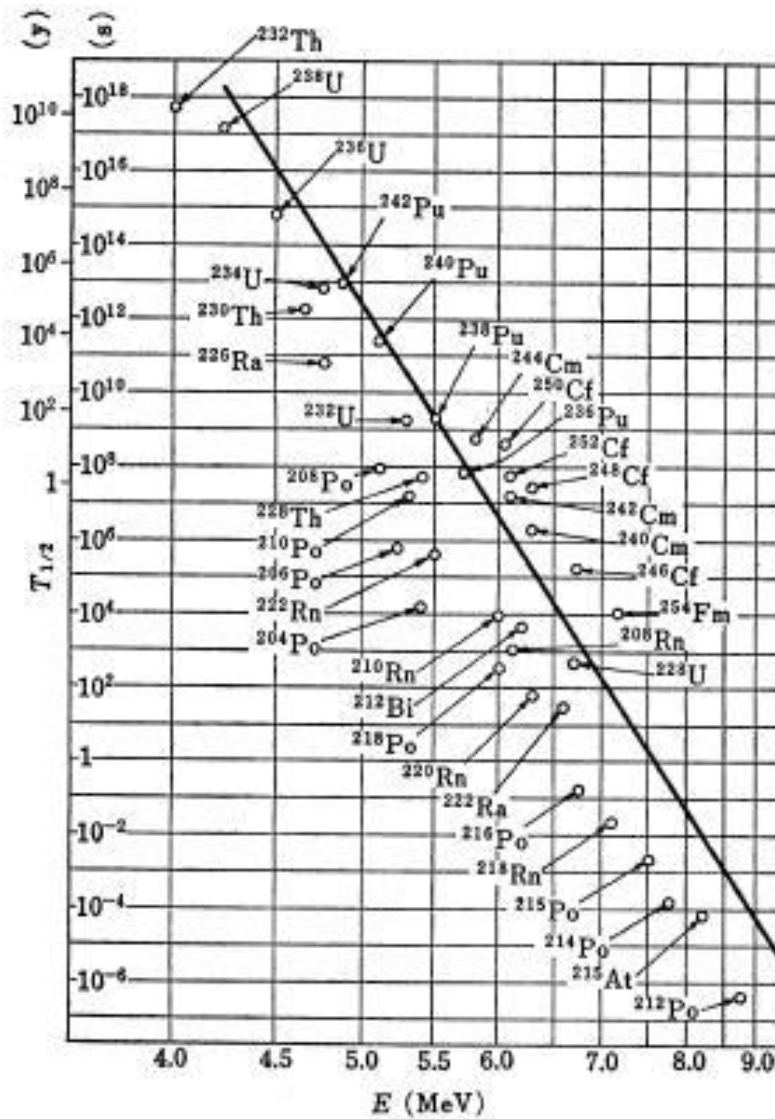

Figure 2: *Test of quantum tunnelling theory of α-decay, eq. (19).*

Appendix A: Estimating the lifetime.

We can make a very rough estimate of the lifetime of a nucleus due to α-decay by imagining the α-particle bound in the nucleus as a de Broglie wave travelling back-and-forth inside the nuclear well of size R. Each time the wave is reflected off the wall at the penetrable barrier it is given an 'opportunity' to tunnel through. This is like a gambler throwing a dice for whom the probability of obtaining a 6 in each throw is 1/6; similarly the probability of tunnelling through each time the de Broglie wave is incident on the barrier is the Gamow factor, e^{-G}. We now need to know how many times per second the de Broglie wave 'attempts' to tunnel. The wave has wavelength λ given by the de Broglie relation:

$$\lambda = \frac{h}{p} = \frac{h}{mv}, \tag{12}$$

which is also approximately the size of the trapping region, the nucleus,

$$\lambda \approx R . \tag{13}$$

Hence, very approximately,

$$v \approx \frac{h}{mR} . \tag{14}$$

Therefore successive encounters with the barrier take a time

$$t_0 \approx \frac{2R}{v} \approx \frac{mR^2}{\pi\hbar} . \tag{15}$$

Now on average we need e^{+G} 'attempts' to tunnel for just one successful tunneling event (just like we need, on average, 6 throws of the dice to achieve one 6). Therefore one tunneling event takes a time

$$\tau \approx t_0 e^{+G} \approx \frac{mR^2}{\pi\hbar} e^{+G} \approx \frac{1}{c}\left(\frac{mR^2}{\pi\hbar c}\right) e^{+G} . \tag{16}$$

For the decay of a nucleus of charge number $Z = Z_1 + Z_2$ into a daughter nucleus of charge number Z_1 and an α-particle of charge number $Z_2 = 2$, the Gamow factor is

$$G(E) \simeq \pi \left(\frac{2m}{\hbar^2}\right)^{\frac{1}{2}} \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right) \frac{1}{\sqrt{E}} \tag{17}$$

where E is the energy of the α-particle; m its mass.